Assignment 1

1. Let $\Sigma = \{0, 1\}$. For every word $w \in \Sigma^*$, let $N_0(w)$ and $N_1(w)$ denote the count of 0's and 1's, respectively, in w. Let L be the language

   $$L = \{ \ w \in \Sigma^* \mid N_0(w) > N_1(w) + 2, \ \text{or} \ N_1(w) > N_0(w) + 2 \ \} .$$

   Prove or disprove whether L is regular.

2. Given languages L_1, L_2 and L_3, each with alphabet Σ, define $L_1/L_2/L_3$ as

   $$L_1/L_2/L_3 = \{ \ w \in \Sigma^* \ \mid \ \exists u \in L_2 \text{ and } \exists v \in L_3, \text{ such that } wuv \in L_1 \ \}$$

   Prove that if L_1 is context-free, and L_2 and L_3 are regular, then $L_1/L_2/L_3$ is context-free.

3. Let L_2 be the language

   $$L = \{ \ \langle M \rangle \ \mid \ \text{there is at least one input string on which the Turing machine } M \text{ does not halt } \}$$

   Here, for a Turing machine M, the notation $\langle M \rangle$ denotes an encoding, over some alphabet, of the code of the Turing machine. Argue, with proof, to which of the following language classes does L belong –

   - Regular
   - Context-free but not Regular.
   - Recursive but not Context-free
   - Recursively enumerable but not recursive
   - Not recursively enumerable

4. Prove that the relation $\leq_p$ is transitive i.e., if languages L_1, L_2, and L_3 are such that

   $$L_1 \leq_p L_2, \text{ and } L_2 \leq L_3$$

   then

   $$L_1 \leq_p L_3.$$

5. Prove that every language in complexity class P is P-complete.