CHAPTER 3

RENORMALIZATION AND THE EVOLUTION OF COUPLING CONSTANTS

In this chapter, we shall derive the one-loop Renormalization-Group Equations (RGE) for the gauge, Yukawa and quartic scalar couplings. The gauge coupling RGE derived are valid for any semisimple Lie group. The Yukawa and quartic scalar coupling RGE on the other hand, depend on the details of the model and the equations derived in this chapter are valid for models with one Higgs-boson doublet and an arbitrary number of fermion generations. (We neglect mixing among quarks.) Finally, the RGE for quartic scalar couplings in a two-doublet model discussed in Chapter 2 are also derived.

From our experience with Quantum Electrodynamics (QED), we know that as soon as one tries to evaluate higher order Feynman graphs involving loops, one faces the problem of divergent integrals. These divergencies could in principle render the theory mathematically inconsistent. However, in certain theories, as in QED, these divergencies appear in a very traceable way, and by a suitable redefinition of the fields and coupling constants, they disappear order by order in perturbation expansion! Such theories are said to be renormalizable and the procedure of removing these unwanted divergencies is called Renormalization.[13]

A renormalizable field theory contains two types of parameters - masses or coupling constants with positive dimensions of mass (i.e. due to $m\bar{\psi}\psi$ or $\lambda\phi^3$ terms in the Lagrangian) and dimensionless coupling constants (i.e. due to $\lambda\phi^4$ and $\bar{\psi}\gamma^{\mu}\psi A_{\mu}$ terms in the Lagrangian). Coupling constants with negative dimensions of mass give cut off, Λ, and then take the limit $\Lambda \to \infty$, in which case the divergencies appear as $\log \Lambda$. Note that in either case, an arbitrary mass parameter has to be introduced which leads to the concept of the running of coupling constants. These ideas will become transparent as we derive the one-loop RGE for gauge coupling constants which we proceed to do now.

3.1. RENORMALIZATION OF GAUGE COUPLING CONSTANTS

Let us consider the case of a simple gauge group G. (Generalization to the case of semisimple gauge groups is straightforward.) The classical Lagrangian density is the same as the one for QCD, given in Eqs.(2.2)-(2.4) of Chapter 2. However, as mentioned there, a gauge covariant quantization requires the addition of a 'gauge fixing term'.[16] This is necessitated by the familiar problem in gauge theories that the gauge invariant Lagrangian does not uniquely determine the gauge field in terms of a source. In QED, this ambiguity in the definition of the photon propagator is overcome by imposing the covariant gauge condition,

$$\partial_\mu A^\mu = 0 . \tag{3.1}$$

Equivalently, one could add a gauge fixing term,

$$L_{g.f} = -\frac{\alpha}{2}(\partial_\mu A^\mu)^2 , \tag{3.2}$$

from which the photon propagator follows:

$$\frac{-i[g^{\mu\nu} + (\alpha-1)p^\mu p^\nu/(p^2 + i\epsilon)]}{(p^2 + i\epsilon)} . \tag{3.3}$$

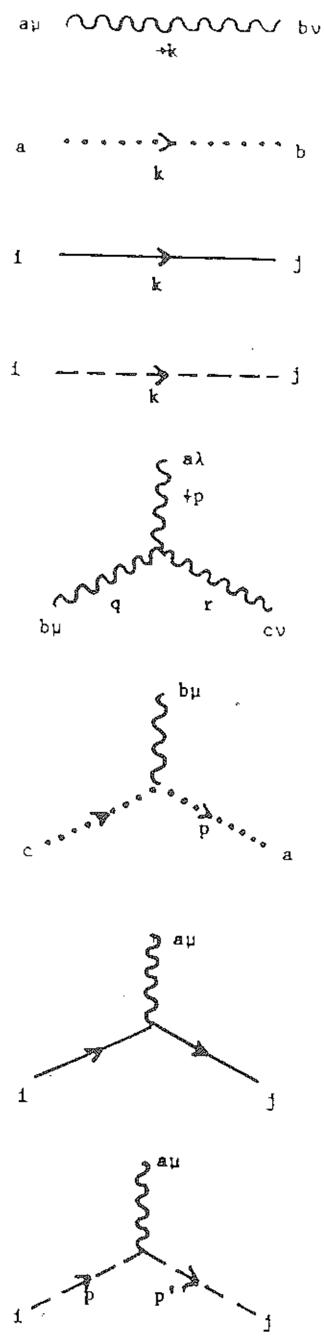
nonrenormalizable theories (eg. Fermi's theory of weak interaction). In this dissertation, we shall be interested in the renormalization of the dimensionless coupling constants only. The asymptotic behavior of the theory is the same as in the massless case, since $m\bar{\psi}\psi$ and $\lambda\phi^3$ terms do not contribute to leading order to the asymptotic expansion of the Green's functions. However, the massless theory does contain a hidden mass parameter, μ, which must be introduced to perform the subtraction necessary to renormalize the theory and render it finite. The subtraction point, μ, is arbitrary. If we change μ, the net effect is to change the value of the coupling constants and the scale of the fields. This leads to the concept of Renormalization-Group Equations and running coupling constants.[14]

Regularization of divergent amplitudes may be achieved in several different ways. An elegant and simple prescription is the dimensional regularization scheme of 't Hooft and Veltman[15] which we shall adopt in this dissertation. This scheme has the advantage that it is explicitly gauge invariant and hence several diagrams do not contribute to the renormalization-group equations. In this scheme, the dimensionality of space-time is lowered from 4 to $n = 4 - \epsilon$ ($\epsilon > 0$), where the integrals are well defined. Then by analytic continuation, one goes to four dimensions, whence the divergencies appear as simple poles in ϵ which can be easily subtracted. It is worthwhile to note that in n dimensions, the gauge couplings are not dimensionless, but have dimensions $(4-n)/2$ in units of mass. (Remember that the action has to be dimensionless.) Similarly, quartic scalar couplings have dimensions $(4-n)$ in units of mass. An alternate procedure is to employ a momentum

Since all the observables are independent of the particular choice of the gauge, the gauge parameter α does not appear in any physical processes. $\alpha = 1$ corresponds to the Feynman gauge and $\alpha = 0$ to the Landau gauge. Feynman gauge is very convenient for the calculation of RGE for the gauge coupling constants, whereas Landau gauge is well suited for the RGE for Yukawa and scalar couplings. The one-loop renormalization-group equations are independent of the gauge chosen, which allows us to work in different gauges for different couplings.

With the gauge fixing term, Eq.(3.2), added to the Lagrangian, the one-loop correction to the gauge boson propagator turns out to be nontranversal (i.e. gauge noninvariant). This is because the gauge fixing term has interfered with the gauge invariance of the theory. This problem is resolved by introducing a set of fictitious scalar fields belonging to the adjoint representation of the group, known as Faddeev-Popov ghost fields,[16] which appear only in closed loops. These particles are known as ghosts, because although spinless, they obey Fermi statistics. Thus each closed ghost loop should be multiplied by a minus sign, like a fermion loop.

The Feynman rules for the effective interactions of gauge bosons with gauge, ghost, fermion and (complex) scalar fields[17] are given in Fig.1. Wavy lines denote gauge fields, dotted lines are for ghost fields, solid lines for fermions and broken lines for scalar fields. There is a factor of -1 for each closed fermion and ghost loops. The non-Abelian character of the theory shows up in the triple gauge boson vertex.



$$\frac{-i\delta^{ab}}{k^2}\left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right)$$

$$\frac{-i\delta^{ab}}{k^2}$$

$$\frac{-i\delta^{ij}}{\not{k}-m}$$

$$\frac{-i\delta^{ij}}{k^2-M^2}$$

$$gf^{abc}((p-q)_\nu g_{\lambda\mu}+(q-r)_\lambda g_{\mu\nu}+(r-p)_\mu g_{\nu\lambda})$$

$$gf^{abc}p^\mu$$

$$-ig\gamma^\mu T^a{}_{ij}$$

$$ig(p+p')^\mu T^a{}_{ij}$$

Fig.1. Feynman rules for gauge boson interactions in a non-Abelian gauge theory.

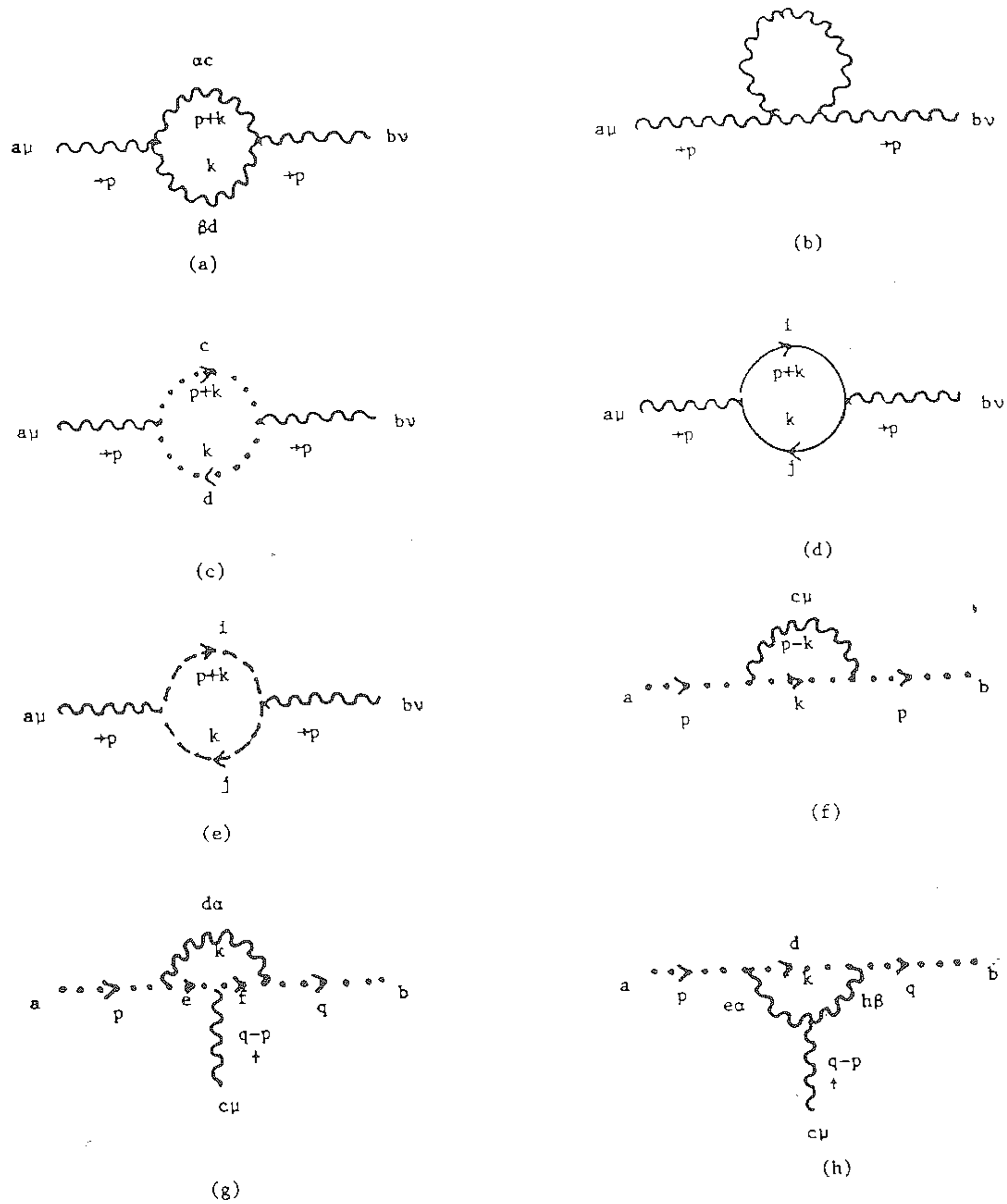


Fig.2. One-loop correction to gauge boson self energy (a-e), ghost self energy (f) and gauge boson-ghost-ghost vertex (g-h).

The renormalizability of non-Abelian gauge theories was established by 't Hooft,[18] first in the context of massless theories which he subsequently generalized to the case of spontaneously broken gauge theories. 't Hooft and Veltman[15] developed the powerful technique of dimensional regularization for the renormalization of such gauge theories, which we shall follow here. The essence of renormalizability is that all infinities appearing in the theory can be removed by absorbing them in a finite number of primitively divergent Green's functions. However, as it turns out, all these primitively divergent Green's functions are not independent, but are related through 'Ward identities'. This is the consequence of the requirement that the renormalized Lagrangian be also gauge invariant. These identities enable us to choose a simple set of primitively divergent graphs for calculating the renormalization-group equations. For the gauge coupling evolution, we choose to evaluate the gauge boson self-energy, ghost self-energy and gauge boson-ghost-ghost vertex correction diagrams.

The gauge boson self-energy diagrams are shown in Figures 2a-2e. In dimensional regularization scheme, all tadpole diagrams vanish.[15] Hence the contribution from Fig.2b is zero. This is because there is no momentum dependence at the vertices, and the diagram could only give a correction to the gauge boson mass. But in a gauge invariant scheme, the gauge boson cannot have a mass. If we adopt some other regularization procedure, Fig.2b will contribute however.

Consider the amplitude for the diagram in Fig.2a. It is

$$M^{ab}_{\mu\nu}(\text{gauge}) = \frac{1}{2}g^2 f^{acd} f^{bcd} \int \frac{d^n k}{(2\pi)^n} \frac{I_{\mu\nu}(k,p)}{k^2(k+p)^2} , \qquad (3.4)$$

where 1/2 is the symmetry factor and

$$I_{\mu\nu}(k,p)=[(k+2p)_{\beta}g_{\mu\alpha}+(k-p)_{\alpha}g_{\mu\beta}-(2k+p)_{\mu}g_{\alpha\beta}]$$
$$[(k+2p)^{\beta}g^{\alpha}_{\nu}+(k-p)^{\alpha}g^{\beta}_{\nu}-(2k+p)_{\nu}g^{\alpha\beta}] \quad . \tag{3.5}$$

Here, we have written the loop integral in n dimension. We shall choose $n = 4 - \epsilon$. The loop integral is clearly divergent in 4 dimensions, but is well defined in n dimensions. When analytically continued to four dimensions, the divergencies appear as simple poles in ϵ. This pole part of the integral can be evaluated using the formulae of Appendix B. First of all, $I_{\mu\nu}(k,p)$ can be expanded to give

$$I_{\mu\nu}(k,p) \;=\; 10k_{\mu}k_{\nu}-2p_{\mu}p_{\nu}+5(p_{\mu}k_{\nu}+k_{\mu}p_{\nu})+g_{\mu\nu}[(k+2p)^2+(k-p)^2]. \tag{3.6}$$

Next, we introduce the Feynman parametrization given in Appendix B.1,

$$\frac{1}{ab} = \int_0^1 dx\, \frac{1}{[a+(b-a)x]^2} \quad . \tag{3.7}$$

The amplitude is then

$$M^{ab}_{\mu\nu}(\text{gauge}) \;=\; \frac{1}{2}g^2 f^{acd} f^{bcd} \int \frac{d^nk}{(2\pi)^n} \int_0^1 dx\, \frac{I_{\mu\nu}(k,p)}{[k^2+x(k^2+2k.p)]^2} \quad . \tag{3.8}$$

Now we shift the momentum variable from k to k+px, without changing the limits of k integration (since k goes from $-\infty$ to $+\infty$) and reverse the order of integration to obtain

$$M^{ab}_{\mu\nu}(\text{gauge}) \;=\; \frac{1}{2}g^2 f^{acd} f^{bcd} \int_0^1 dx \int \frac{d^nk}{(2\pi)^n}\, \frac{I_{\mu\nu}(k+px,p)}{[k^2+p^2x(1-x)]^2} \quad , \tag{3.9}$$

where

$$I_{\mu\nu}(k+px,p) = 10k_{\mu\nu} - 2p_\mu p_\nu[1+5x(1-x)] + g_{\mu\nu}[2k^2+p^2(5+x^2-2x)] \ . \qquad (3.10)$$

Here we have dropped the terms odd in k, since when integrated they yield zéro. The k integration is performed using Eqs. B6-B8 of Appendix B:

$$\int \frac{d^nk}{(2\pi)^n} \frac{1}{[k^2+p^2x(1-x)]^2} = \frac{i}{16\pi^2}\Gamma(\tfrac{\epsilon}{2}) = \frac{i}{16\pi^2}\frac{2}{\epsilon} \qquad (3.11)$$

$$\int \frac{d^nk}{(2\pi)^n} \frac{k^2}{[k^2+p^2x(1-x)]^2} = \frac{i}{16\pi^2}2\Gamma(-1+\tfrac{\epsilon}{2})p^2x(1-x) = \frac{-i}{16\pi^2}2p^2x(1-x)\frac{2}{\epsilon} \qquad (3.12)$$

$$\int \frac{d^nk}{(2\pi)^n} \frac{k_\mu k_\nu}{[k^2+p^2x(1-x)]^2} = \frac{i}{16\pi^2}\Gamma(-1+\tfrac{\epsilon}{2})p^2x(1-x)g_{\mu\nu} = \frac{-i}{16\pi^2}p^2x(1-x)g_{\mu\nu}\frac{2}{\epsilon} \qquad (3.13)$$

The second halves of Eqs.(3.11)-(3.13) are obtained using the small ϵ expansion of the Γ function given in Appendix B.10. We have kept only the pole part of the expressions, since in a mass independent renormalization scheme, which we choose to work in, the finite part of the amplitudes can be set to zero. Using these expressions and performing the elementary integration over x, Eq. (3.9) gives

$$M^{ab}_{\mu\nu}(\text{gauge}) = \frac{-ig^2}{16\pi^2} f^{acd}f^{bcd}(\tfrac{11}{6}p_\mu p_\nu - \tfrac{19}{12}p^2g_{\mu\nu})\frac{2}{\epsilon} + \ldots \qquad (3.14)$$

Note that this expression is not gauge invariant - i.e. not proportional to $(p_\mu p_\nu - p^2 g_{\mu\nu})$.

The amplitude for the ghost contribution to the gauge boson self-energy (Fig.2c) is

$$M^{ab}_{\mu\nu}(\text{ghost}) = -g^2f^{acd}f^{bcd}\int \frac{d^nk}{(2\pi)^n} \frac{(p+k)_\mu k_\nu}{k^2(p+k)^2} \ , \qquad (3.15)$$

where the minus sign is due to the closed ghost loop. This integral can be performed exactly the same way as before - introducing Feynman parametrization, Eq.(3.7), shifting the momentum variable to k+px and using Eqs.(3.11) and (3.13), with the result

$$M^{ab}_{\mu\nu}(\text{ghost}) = \frac{-ig^2}{16\pi^2} f^{acd} f^{bcd} \left(\frac{1}{6} p_\mu p_\nu + \frac{p^2}{12} g_{\mu\nu}\right) \frac{2}{\epsilon} + \ldots\ldots \tag{3.16}$$

Note that the sum of the gauge and ghost contributions,

$$M^{ab}_{\mu\nu}(\text{gauge+ghost}) = \frac{-ig^2}{16\pi^2} C_2(G)\delta^{ab} \frac{5}{3} (p_\mu p_\nu - p^2 g_{\mu\nu})\frac{2}{\epsilon} \tag{3.17}$$

is gauge invariant. Here we have made use of the group theoretic identity given in Eq.(A.4) of Appendix A.

The fermion contribution, Fig.2d, is

$$M^{ab}_{\mu\nu}(\text{fermion}) = -g^2 T^a{}_{ij} T^b{}_{ji} \int \frac{d^n k}{(2\pi)^n} \frac{\mathrm{Tr}[\gamma_\mu(\not{p}+\not{k}+m)\gamma_\mu(\not{k}+m)]}{(k^2-m^2)[(p+k)^2-m^2]} . \tag{3.18}$$

The Trace in the numerator of (3.18) gives

$$\mathrm{Tr}[] = 4[2k_\mu k_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(k^2 + p.k - m^2)] . \tag{3.19}$$

Strictly speaking, the γ matrix algebra is different in $4-\epsilon$ dimensions, but this difference will show up only in the finite part of the amplitude, which we are setting to zero. Using the now familiar techniques, and the group theoretic identity, Eq.(A.5) of Appendix A,

$$M^{ab}_{\mu\nu}(\text{fermion}) = \frac{ig^2}{16\pi^2} T(R)\delta^{ab} \frac{4}{3}(p_\mu p_\nu - p^2 g_{\mu\nu})\frac{2}{\epsilon} + \ldots\ldots \tag{3.20}$$

Finally, the complex scalar contribution to the gauge boson self energy is

$$M^{ab}_{\mu\nu}(\text{scalar}) = ig^2 T^a{}_{ij} T^b{}_{ji} \int \frac{d^n k}{(2\pi)^n} \frac{(2k+p)_\mu (2k+p)_\nu}{k^2 (p+k)^2} , \qquad (3.21)$$

which leads to

$$M^{ab}_{\mu\nu}(\text{scalar}) = \frac{ig^2}{16\pi^2} T(S)\delta^{ab} \frac{1}{3}(p_\mu p_\nu - p^2 g_{\mu\nu})\frac{2}{\epsilon} + \ldots . \qquad (3.22)$$

Hence the total contribution to the gauge boson propagator is

$$M^{ab}_{\mu\nu}(\text{total}) = \frac{-ig^2}{16\pi^2}(p_\mu p_\nu - p^2 g_{\mu\nu})[\tfrac{5}{3}C_2(G) - \tfrac{4}{3}T(R) - \tfrac{1}{3}T(S)]\frac{2}{\epsilon} + \ldots (3.23)$$

Next consider the ghost self-energy diagram. Only the gauge bosons contribute to this, (Fig.2f), with an amplitude given by

$$\Pi^{ab}(p) = g^2 f^{acd} f^{bcd} \int \frac{d^n k}{(2\pi)^n} \frac{(p-k)_\mu p_\nu g_{\mu\nu}}{k^2 (p-k)^2} . \qquad (3.24)$$

Using the same techniques as for the gauge boson self-energy, the amplitude is evaluated to be

$$\Pi^{ab}(p) = \frac{ig^2}{16\pi^2} C_2(G)\delta^{ab} \frac{p^2}{2} \frac{2}{\epsilon} + \ldots . \qquad (3.25)$$

Finally, the ghost-ghost-vector boson vertex is corrected by the two diagrams shown in Figures 2g and 2h. The amplitude for the first is

$$\Pi^{abc}_{\mu 1}(p,q) = ig^3 f^{eda} f^{hce} f^{bdh} \int \frac{d^n k}{(2\pi)^n} \frac{(q-k)_\mu (p-k).q}{k^2 (p-k)^2 (q-k)^2} . \qquad (3.26)$$

The divergent part of this integral is the $k_\mu k_\nu$ term only. It can be evaluated by introducing the Feynman parametrization, Eq.(B.2) of Appendix B, shifting the integration variable to k-px-qy and again dropping the finite part of the integral. The result is

$$\Pi^{abc}_{\mu 1}(p,q) = \frac{g^3}{16\pi^2} f^{aed} f^{bdh} f^{che} \frac{q_\mu}{4} \frac{2}{\epsilon} + \ldots\ldots \qquad (3.27)$$

The group theoretical factor can be simplified using Eq.(A.11) of Appendix A:

$$\Pi^{abc}_{\mu 1}(p,q) = \frac{g^3}{16\pi^2} C_2(G) f^{abc} q_\mu \frac{1}{8} \frac{2}{\epsilon} + \ldots\ldots \qquad (3.28)$$

The diagram Fig.2h can be calculated to be

$$\Pi^{abc}_{\mu 2} = ig^3 f^{dae} f^{bhd} f^{ceh} \int \frac{d^n k}{(2\pi)^n} \frac{k_\alpha q_\beta [(q+k-2p)^\beta g^\alpha_\mu + (p+q-2k)_\mu g^{\alpha\beta} + (k+2p-q)^\alpha g^\beta_\mu]}{k^2 (q-k)^2 (p-k)^2} \qquad (3.29)$$

The divergent part of Eq.(3.29) is

$$\Pi^{abc}_{\mu 2}(p,q) = ig^3 f^{dea} f^{bhd} f^{ceh} \int \frac{d^n k}{(2\pi)^n} \frac{(k^2 q_\mu - k_\mu k_\nu q_\nu)}{k^2 (k-q)^2 (k-p)^2} , \qquad (3.30)$$

which evaluates to

$$\Pi^{abc}_{\mu 2}(p,q) = \frac{g^3}{16\pi^2} C_2(G) f^{abc} q_\mu \frac{3}{8} \frac{2}{\epsilon} + \ldots\ldots \qquad (3.31)$$

Hence the total correction to the ghost-ghost-vector boson vertex is

$$\Pi^{abc}_{\mu}(p,q) = \Pi^{abc}_{\mu 1}(p,q) + \Pi^{abc}_{\mu 2}(p,q) = \frac{g^3}{16\pi^2} C_2(G) f^{abc} \frac{1}{2} \frac{2}{\epsilon} . \qquad (3.32)$$

We are now in a position to perform the renormalization of the gauge coupling constants. First consider the one-loop corrected gauge boson propagator. In Feynman gauge, with our result Eq.(3.23), it is

$$\frac{-ig_{\mu\nu}\delta^{ab}}{p^2} + \frac{(-i)}{p^2} M^{ab}_{\mu\nu}(p^2) \frac{(-i)}{p^2} \qquad (3.33)$$

$$= \frac{-ig_{\mu\nu}\delta^{ab}}{p^2}\left[1 + \frac{g^2}{16\pi^2}\{\tfrac{5}{3}C_2(G) - \tfrac{4}{3}T(R) - \tfrac{1}{3}T(S)\}\tfrac{2}{\epsilon}\right]. \qquad (3.34)$$

Here we have considered only the longitudinal part of Eq.(3.23), since the transverse part contributes to the redefinition of the gauge fixing term. From Eq.(3.34), it is clear that the infinity associated with the vector boson propagator can be absorbed into the redefinition of the wave function. The wave function renormalization constant, Z_3, is

$$Z_3 = 1 + \frac{g^2}{16\pi^2}\left[\tfrac{5}{3}C_2(G) - \tfrac{4}{3}T(R) - \tfrac{1}{3}T(S)\right]\tfrac{2}{\epsilon}\ . \qquad (3.35)$$

Similarly, from Eq.(3.25), the ghost wave function renormalization constant is

$$Z_3' = 1 + \frac{g^2}{16\pi^2}\, C_2(G)\frac{1}{2}\,\frac{2}{\epsilon}\ . \qquad (3.36)$$

The ghost-vector charge renormalization constant, Z_1' is the ratio of the bare to the unrenormalized three-point function, $\Pi_\mu^{abc}(p,q)$, which from Eq.(3.32) follows:

$$Z_1' = 1 - \frac{g^2}{16\pi^2}\,\frac{1}{2}C_2(G)\frac{2}{\epsilon} \qquad (3.37)$$

The analogous vector charge renormalization constant, Z_1, can be obtained without any more work, using the Ward identity,

$$\frac{Z_1}{Z_1'} = \frac{Z_3}{Z_3'}\ , \qquad (3.38)$$

which yields,

$$Z_1 = 1 + \frac{g^2}{16\pi^2}[\tfrac{2}{3}C_2(G) - \tfrac{4}{3}T(R) - \tfrac{1}{3}T(S)]\frac{2}{\epsilon} \quad . \tag{3.39}$$

Note that the renormalized charge, scale of the fields and gauge parameter depend on the renormalization point, μ, which is arbitrary. However, the unrenormalized Lagrangian has no dependence on μ. The fact that the unrenormalized couplings are independent of μ is expressed through the renormalization-group equation,[14]

$$\mu \frac{\partial}{\partial\mu}\Gamma_u^{(n)}(\mu, g_u, \alpha_u) = 0 \ , \tag{3.40}$$

where Γ is the unrenormalized one particle-irreducible Green's function. Using the chain rule of differentiation,

$$[\mu \frac{\partial}{\partial\mu} + \beta(g,\alpha)\frac{\partial}{\partial g} - n\gamma(g,\alpha) + \delta(g,\alpha)\frac{\partial}{\partial\alpha}]\Gamma^{(n)}(g,\alpha,\mu) = 0 \ , \tag{3.41}$$

where

$$\beta(g,\alpha) = \mu\frac{\partial g}{\partial\mu} \ , \quad \gamma(g,\alpha) = \frac{\mu}{2}\frac{\partial}{\partial\mu}\ln Z_3 \ , \quad \delta(g,\alpha) = \mu\frac{\partial\alpha}{\partial\mu} \quad . \tag{3.42}$$

The bare and renormalized gauge couplings are related by

$$g_0 = \frac{Z_1}{Z_3^{3/2}}\, g \ . \tag{3.43}$$

The 3/2 power is because only the square root of the wave function renormalization appears in the coupling constant renormalization, and there are three external vector boson lines in the definition of g. Remembering that in 4-ϵ dimension, gauge couplings have dimensions of $(\epsilon/2)\mu$, the β function is found to be

$$\beta(g,\alpha) = -\frac{g^3}{16\pi^2}[\frac{11}{3}C_2(G) - \frac{4}{3}T(R) - \frac{1}{3}T(S)] . \qquad (3.44)$$

In general, the fermions and scalars will belong to reducible representation of the group, in which case, one has to sum over their contributions. For semisimple groups, Eq.(3.44) is trivially modified as follows: Consider a semisimple group $G_1 \times G_2$. Then the beta function for g_1 is[19]

$$\beta_{g_1} = -\frac{g_1^3}{16\pi^2}[\frac{11}{3}C_2(G_1) - \frac{4}{3}T(R_1)d(R_2) - \frac{1}{3}T(S_1)d(S_2)], \qquad (3.45)$$

where $d(R_2)$ is the dimensionality of the fermion representation under G_2, etc. We wish to remind the reader that we have assumed the Higgs-scalars to be complex. For real representation, the factor 1/3 should be replaced by 1/6 in Eq.(44)-(45). For chiral fermion representations,(i.e. if the left and right-handed fermions transform differently), as is the case in the Standard Model, the factor 4/3T(R) should be replaced by 2/3T(R).

As an example, let us work out the beta functions for the Standard Model with n_G generations of fermions and n_H Higgs-boson doublets. For the $SU(3)_C$ coupling, Higgs-bosons do not contribute. Since left-handed and right-handed quarks are triplets under $SU(3)_C$, $T(R) = 1/2$. Hence the β function for g_3 is

$$\beta_{g_3} = -\frac{g_3^3}{16\pi^2}[\frac{11}{3}.3 - \frac{4}{3}.(\frac{1}{2}).2.n_G] = -\frac{g_3^3}{16\pi^2}(11- \frac{4}{3}n_G). \qquad (3.46)$$

For the $SU(2)_L$ coupling,

$$\beta_{g_2} = - \frac{g_2^3}{16\pi^2}\left[\frac{11}{3}.2 - \frac{2}{3}\left(\frac{1}{2}.3+\frac{1}{2}\right)n_g - \frac{1}{3}\,\frac{1}{2}n_H\right] = - \frac{g_2^3}{16\pi^2}\left(\frac{22}{3} - \frac{4}{3}n_G - \frac{1}{6}n_H\right). \quad (3.47)$$

From the hypercharge assignment given in Table 1 of Chapter 2, the corresponding equation for the U(1) gauge coupling is

$$\beta_{g_1} = \frac{g_1^3}{16\pi^2}\left[\,\frac{2}{3}\left(\frac{1}{4}.2+1+\frac{1}{36}2.3+\frac{4}{9}.3+\frac{1}{9}.3\right)n_G + \frac{1}{3}\left(\frac{1}{4}.2\right)n_H\right] = \frac{g_1^3}{16\pi^2}\left(\frac{20}{9}n_G + \frac{1}{6}n_H\right). \quad (3.48)$$

The $SU(3)_C$ and $SU(2)_L$ couplings decrease with momentum, as long as the number of generations and Higgs doublets are not too large, (asymptotic freedom), whereas the U(1) coupling increases with momentum.

3.2. RENORMALIZATION OF YUKAWA AND QUARTIC SCALAR COUPLINGS

Since the fermions and Higgs-bosons will belong to reducible representations of the group in general, the renormalization of their couplings depends on the details of the model. In the following, we shall derive the RGE for Yukawa and quartic scalar couplings in the Standard Model with one or two Higgs-boson doublets and an arbitrary number of fermion generations.[20]

It is very convenient to work in the Landau gauge for this purpose. Several diagrams do not contribute to the beta functions in this gauge, since they give rise to convergent integrals. The problem of evaluating the beta function can be reduced to one of mere combinatorics, if we evaluate the contribution of each model-independent loop factor to the beta function. In Fig.3, we list these loop factors in Landau gauge. The ones not listed give rise to convergent diagrams. For example, the W boson exchange diagrams for the fermion Yukawa couplings do not contribute to the beta functions in Landau gauge. The factors listed in Fig.3 have an overall $1/(8\pi^2)$ and a factor 1/2 for each identical internal lines. These factors are obtained in a manner analogous to the gauge coupling renormalization - by identifying the coefficient of the pole term of the amplitude. These pole terms are simply related to the coefficients of the beta function. Gauge boson correction to the scalar propagator should be divided by 2 since it is the square root of the wave function renormalization that appears in the beta function. In Fig.3, such factors are already included. As an illustrative example, we calculate the loop factor for the gauge boson correction to the Yukawa couplings.

For this diagram, no spurious singularities arise if we set the external momenta of the quarks and Higgs-boson to zero. Similarly, all the masses can be set to zero, since we are not interested in the renormalization of masses. Then the model independent part of the amplitude for the gauge boson correction diagram to the Yukawa coupling is

$$\bar{u}_L \int \frac{d^n k}{(2\pi)^n} \; (-i\gamma^\mu) \frac{(-i)}{\not{k}} \frac{(-i)}{(-\not{k})} \; (-i\gamma^\nu) \; \frac{(-i)}{k^2} (g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}) u_R \qquad (3.49)$$

$$= \bar{u}_L \int \frac{d^n k}{(2\pi)^n} \frac{(i)(4-1)}{k^4} u_R \qquad (3.50)$$

$$= \bar{u}_L u_R [\frac{3i}{16\pi^2}] i\Gamma(\frac{\epsilon}{2}) = - \frac{3}{8\pi^2} \frac{1}{\epsilon} \bar{u}_L u_R \; . \qquad (3.51)$$

Hence the corresponding factor is -3, as shown in Fig.3.

Consider first one generation of fermions in the Standard Model

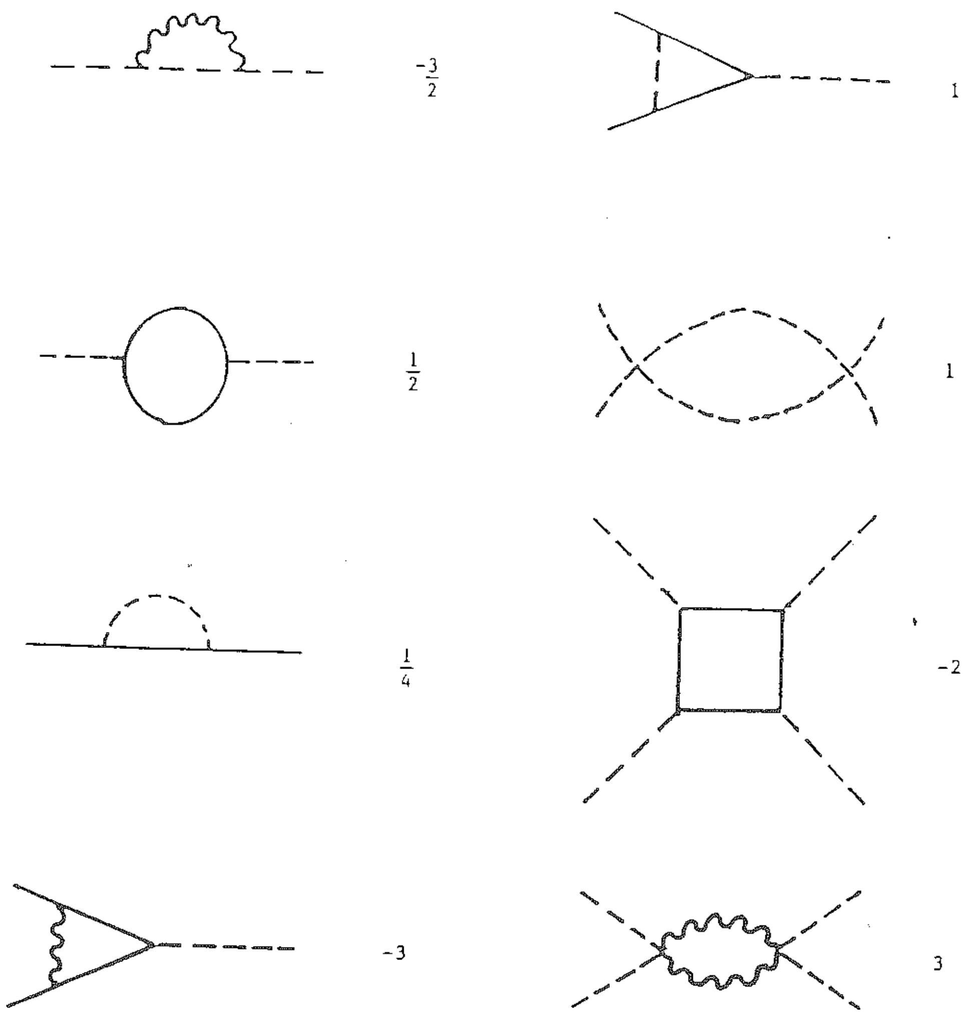


Fig.3. Loop factors for the evaluation of β functions for Yukawa and quartic scalar couplings in the Landau gauge. There is an overall multiplicative factor $1/8\pi^2$ and a fator 1/2 for identical internal lines.

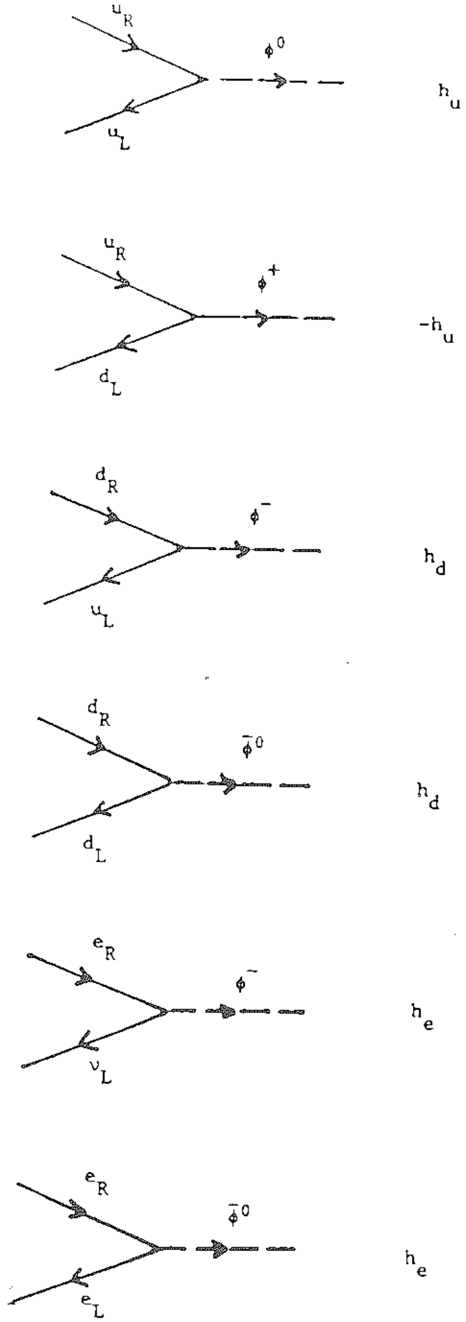


Fig.4. Feynman rules for fermion-Higgs-boson interactions in the Standard Model with one generation.

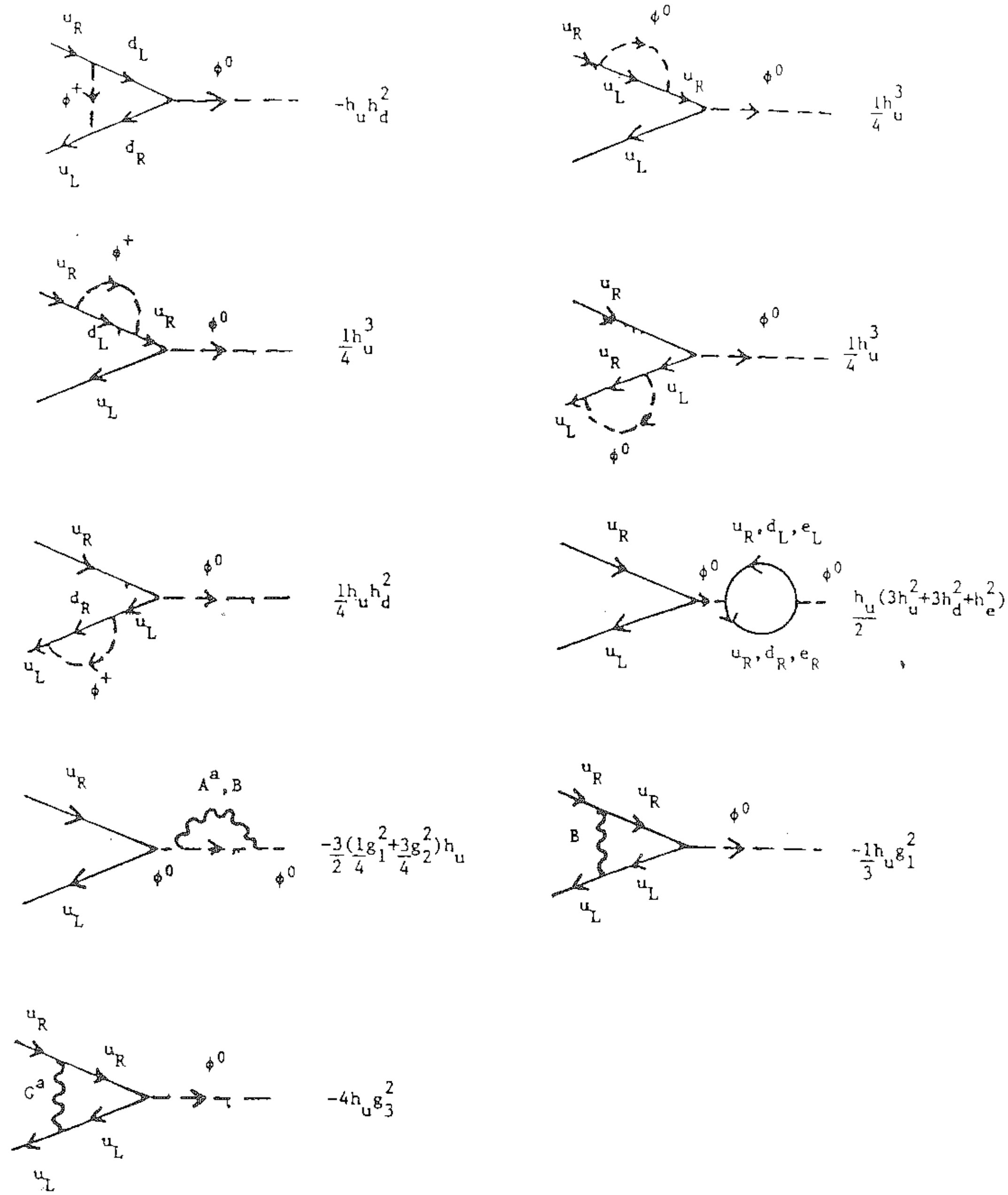


Fig.5. One-loop correction to the Yukawa coupling h_u and the corresponding contribution to the β function in Landau gauge. There is an overall factor $1/8\pi^2$.

that couples to a single Higgs-boson doublet. The Yukawa coupling is

$$L_Y = h_u[\bar{u}_L u_R \bar{\phi}^0 - \bar{d}_L u_R \phi^-] + h_d[\bar{u}_L d_R \phi^+ + \bar{d}_L d_R \phi^0]$$

$$+ h_e[\bar{\nu}_L e_R \phi^+ + \bar{e}_L e_R \phi^0] + \text{H.C.} \qquad (3.52)$$

The associated Feynman rules are shown in Fig.4 and the one loop diagrams correcting the h_u vertex are worked out in Fig.5. Again each diagram has an overall $1/(8\pi^2)$ multiplying it. The loop factors given in Fig.3 are made use of in evaluating the factors of Fig.5. As is clear from here, the problem of evaluating the beta function reduces to one of combinatorics. For the group theoretical factors, we use the results of Appendix A. Summing over all the diagrams of Fig.5, we have

$$8\pi^2 \frac{dh_u^2}{dt} = h_u^2[\frac{9}{2}h_u^2 + \frac{3}{2}h_d^2 + h_e^2 - \frac{17}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2] \ . \qquad (3.53)$$

Here we have defined $t=\log(\mu)$. The evolution of h_d and h_e can be easily obtained without any more work. For h_d, all one-loop diagrams are the same as for h_u, except for the U(1) gauge boson correction, since the d and u quarks differ in their hypercharge. The factor for this diagram should be replaced by +1/6, instead of -1/3. Similarly, for h_e, this factor should be -3/2. Moreover, the gluon correction diagram is absent for h_e. Consequently,

$$8\pi^2 \frac{dh_d^2}{dt} = h_d^2[\frac{9}{2}h_d^2 + \frac{3}{2}h_u^2 + h_e^2 - \frac{5}{12}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2] \ , \qquad (3.54)$$

$$8\pi^2 \frac{dh_e^2}{dt} = h_e^2[\frac{5}{2}h_e^2 + 3h_u^2 + 3h_d^2 - \frac{15}{4}g_1^2 - \frac{9}{4}g_2^2] \ . \qquad (3.55)$$

Clearly, Eqs.(3.53)-(3.55) are valid for an arbitrary number of fermion generations coupled to a single Higgs-boson doublet (assuming negligible quark mixings).

Now consider the evolution of the quartic scalar coupling, λ, in a one Higgs doublet (minimal) Standard Model. The classical Higgs potential is

$$V(\phi) = \mu^2\phi^\dagger\phi + \frac{\lambda}{2}(\phi^\dagger\phi)^2 , \tag{3.56}$$

and the gauge boson kinetic energy term is

$$(D_\mu\phi)^\dagger(D_\mu\phi) = \left|(\partial_\mu - \frac{ig_1B_\mu}{2} - \frac{ig_2\tau^aA^a_\mu}{2})\phi\right|^2 , \tag{3.57}$$

as defined in Chapter 2. The associated Feynman rules are given in Fig.6. Note the presence of symmetry factors. Thus the four point charged Higgs vertex has a factor 2 for identical ϕ^+, and another factor 2 for identical ϕ^-. From these Feynman rules, one arrives at the one-loop diagrams for the (2λ) vertex in Fig.7. These diagrams can be thought of as two processes:

$$\phi^+\phi^- \to \phi^+\phi^- , \tag{3.58}$$

$$\phi^+\phi^+ \to \phi^+\phi^+ . \tag{3.59}$$

For the first process, Eq(3.58), if we interchange the momenta of ϕ^+ in the initial and final states, we obtain a topologically inequivalent diagram, which is denoted by + crossed in Fig.7. (Any more interchange gives topologically equivalent diagrams). For identical internal lines, a symmetry factor 1/2 has been put in. Again only one generation of

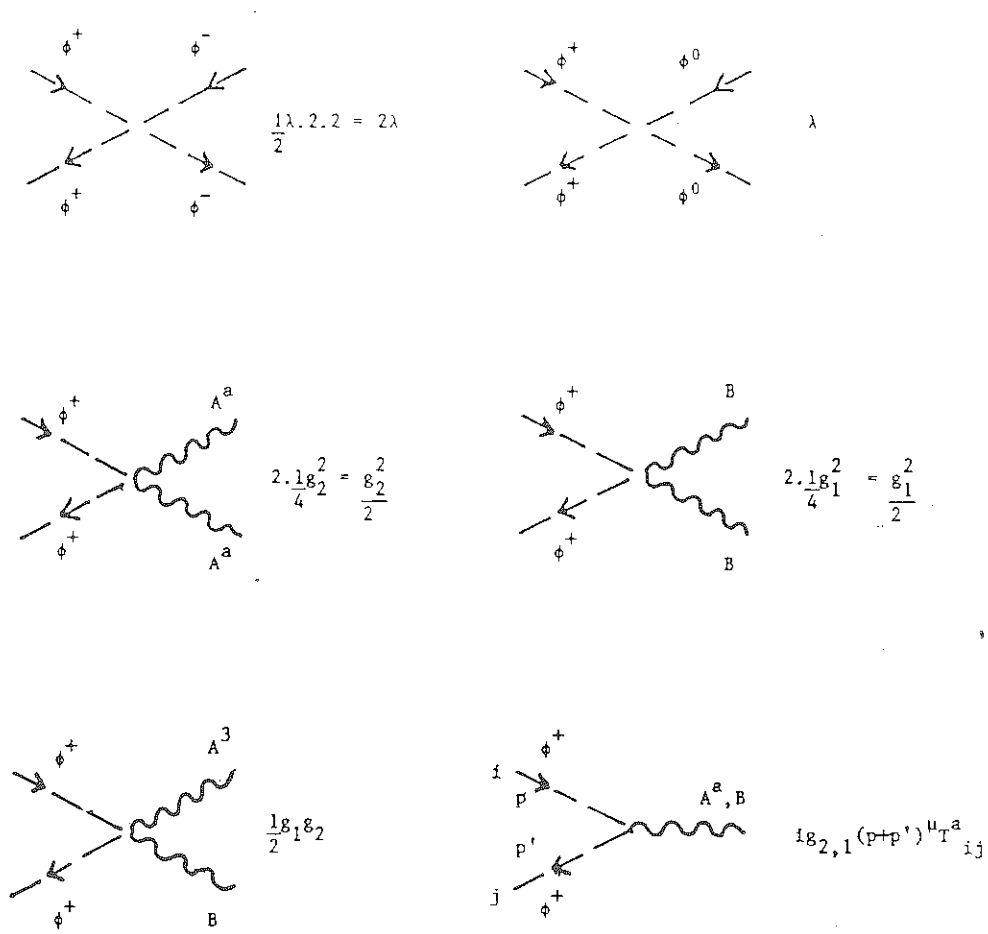


Fig.6. Feynman rules for the Higgs-boson interactions in the Standard Model with one Higgs-boson doublet.

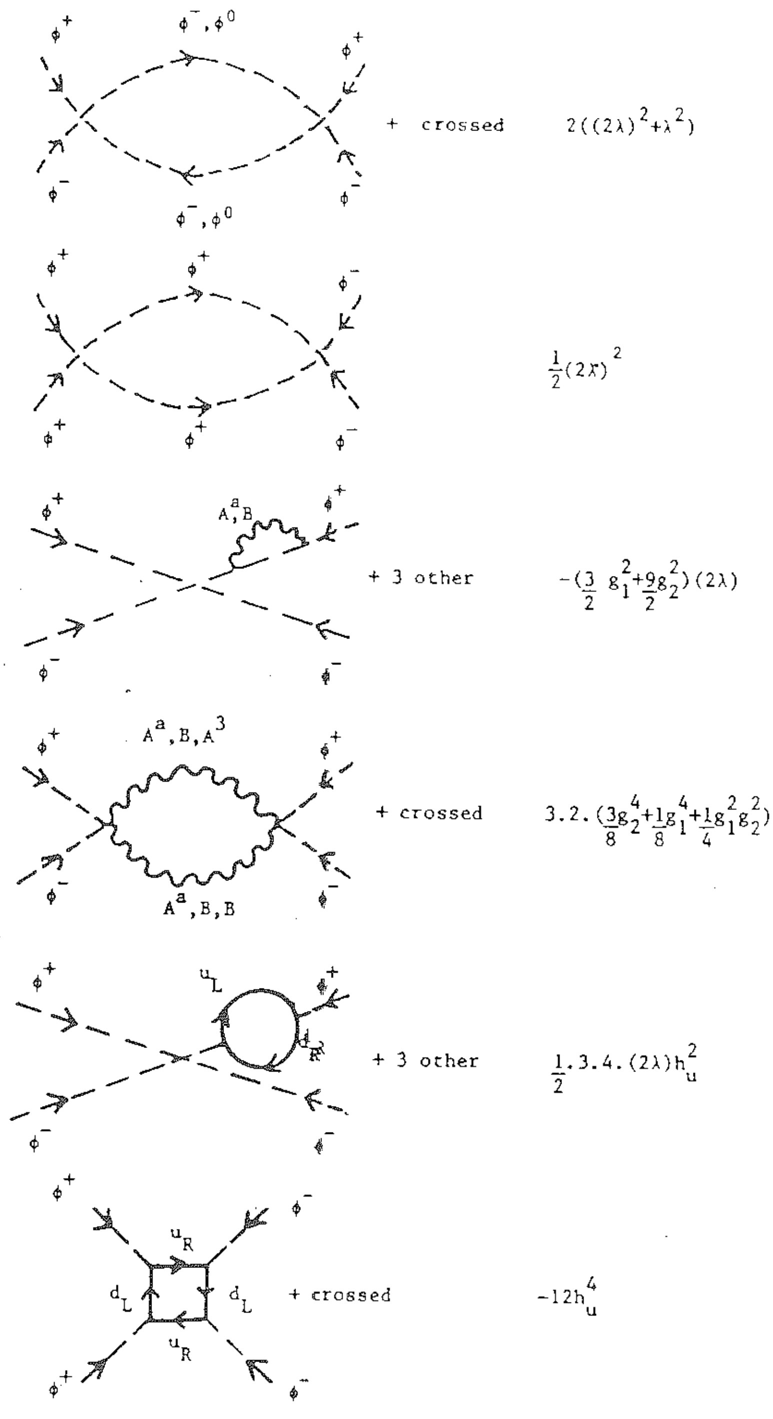


Fig.7. One-loop corrections to the vertex (2λ) in Landau gauge. Each diagram should be multiplied by a factor $1/8\pi^2$.

fermions has been considered, which can be easily generalized to an arbitrary number of generations. Keeping in mind the case of a heavy top quark, we keep only those terms proportional to h_u. Summing over the diagrams given in Fig.7, we arrive at

$$8\pi^2\frac{d\lambda}{dt} = 6\lambda^2 - (\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2)\lambda + \frac{3}{8}g_1^4 + \frac{9}{8}g_2^4 + \frac{3}{4}g_1^2g_2^2 + 6h_u^2\lambda - 6h_u^4 \quad . \quad (3.60)$$

Finally, we consider the evolution of quartic scalar couplings in a two-doublet model discussed in Chapter 2. The classical potential is given in Eq.(2.41) of Chapter 2, from which the Feynman rules listed in Figures 8-9 follow. The interaction of Higgs-bosons with gauge bosons is the same as in the one doublet model, but now with both doublets. In Figures 10-11, the Higgs-boson contributions to the five quartic scalar couplings are evaluated (a-b are for $2\lambda_1$, c-d for $2\lambda_2$, e-g for λ_3, h-j for λ_4 and k-l for $2\lambda_5$). Fig.12 summarizes the gauge contributions to the evolution of these couplings and finally Fig.13 the heavy up quark (which couples only to ϕ_1) contribution. Summing over these graphs, one arrives at the full renormalization-group equations for λ_1-λ_5:

$$8\pi^2\frac{d\lambda_1}{dt} = 6\lambda_1^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + \lambda_5^2 - \lambda_1(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2) + \frac{3}{8}g_1^4 + \frac{3}{4}g_1^2g_2^2 + \frac{9}{8}g_2^4 + 6\lambda_1h_u^2 - 6h_u^4 \ , \quad (3.61)$$

$$8\pi^2\frac{d\lambda_2}{dt} = 6\lambda_2^2 + 2\lambda_2^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + \lambda_5^2 - \lambda_2(\frac{3}{2}g_1^2 + \frac{9}{2}g_2^2) + \frac{3}{8}g_1^4 + \frac{3}{4}g_1^2g_2^2 + \frac{9}{8}g_2^4 \ , \quad (3.62)$$

$$8\pi^2\frac{d\lambda_3}{dt} = (\lambda_1+\lambda_2)(3\lambda_3+\lambda_4) + 2\lambda_3^2+\lambda_4^2+\lambda_5^2-\lambda_3\left(\frac{3}{2}g_1^2+\frac{9}{2}g_2^2\right)+\frac{3}{8}g_1^4 - \frac{3}{4}g_1^2g_2^2+\frac{9}{8}g_2^4+3\lambda_3h_u^2 \quad , \tag{3.63}$$

$$8\pi^2\frac{d\lambda_4}{dt} = (\lambda_1+\lambda_2)\lambda_4+2\lambda_4(2\lambda_3+\lambda_4)+4\lambda_5^2-\lambda_4\left(\frac{3}{2}g_1^2+\frac{9}{2}g_2^2\right)+\frac{3}{2}g_1^2g_2^2 + 3\lambda_4h_u^2 \ , \tag{3.64}$$

$$8\pi^2\frac{d\lambda_5}{dt} = \lambda_5\left[\lambda_1+\lambda_2+4\lambda_3+6\lambda_4-\frac{3}{2}g_1^2-\frac{9}{2}g_2^2+3h_u^2\right] \ . \tag{3.65}$$

The structure of these renormalization-group equations will be fully exploited to find functional relationships among various couplings and to bound these couplings in Chapter 7.

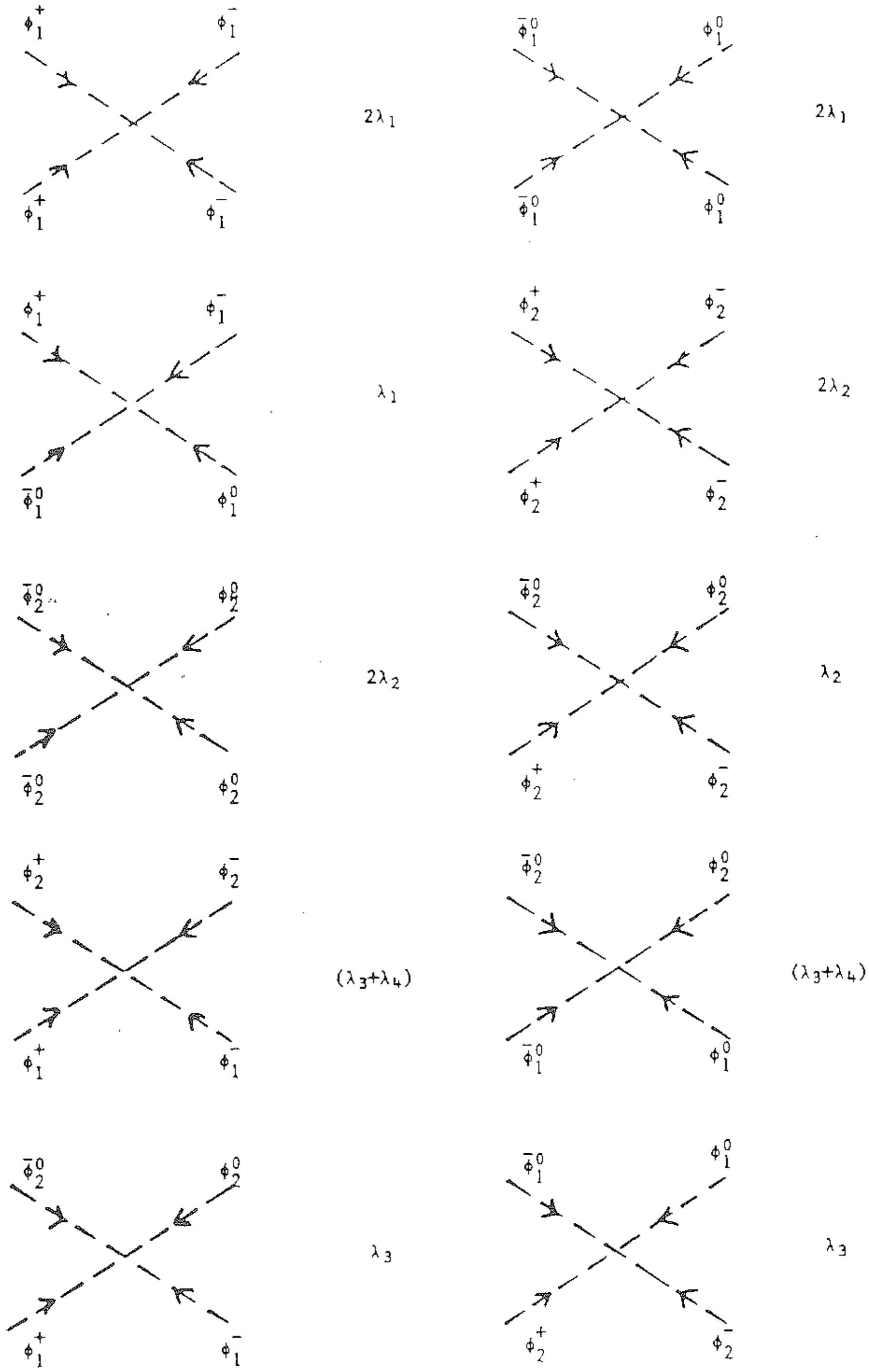


Fig.8. Feynman rules for Higgs-boson self interaction in a two-doublet model.

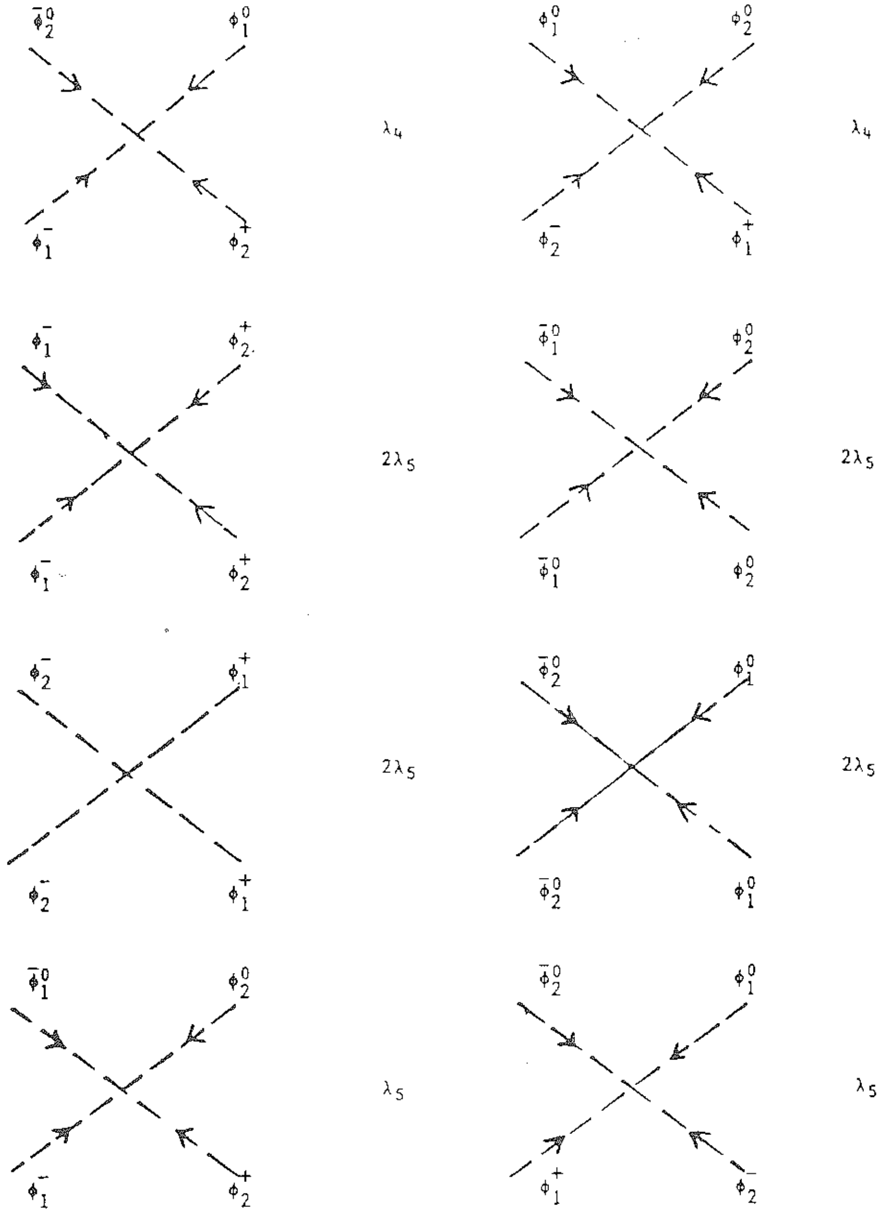
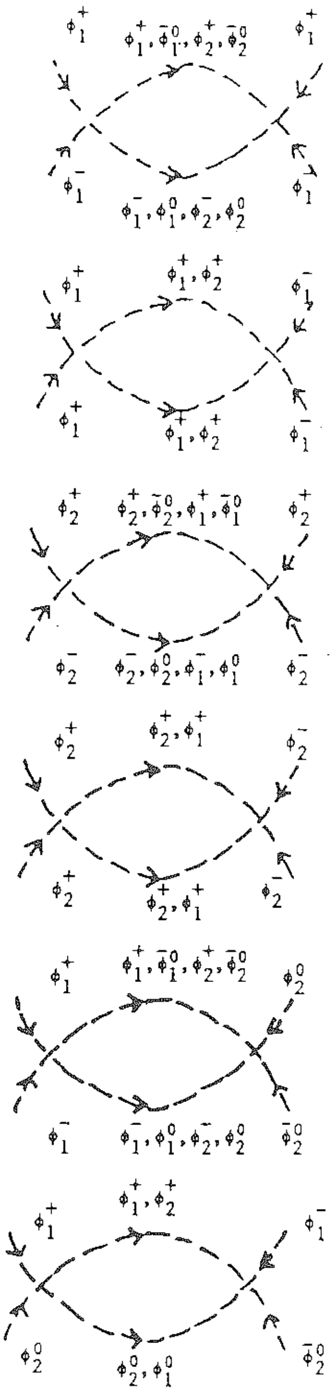


Fig.9. Feynman rules for Higgs-boson self interaction in a two-doublet model.



(a) + crossed $\quad 2((2\lambda_1)^2+\lambda_1^2+(\lambda_3+\lambda_4)^2+\lambda_3^2)$

(b) $\quad \frac{1}{2}(2\lambda_1)^2 + \frac{1}{2}(2\lambda_5)^2$

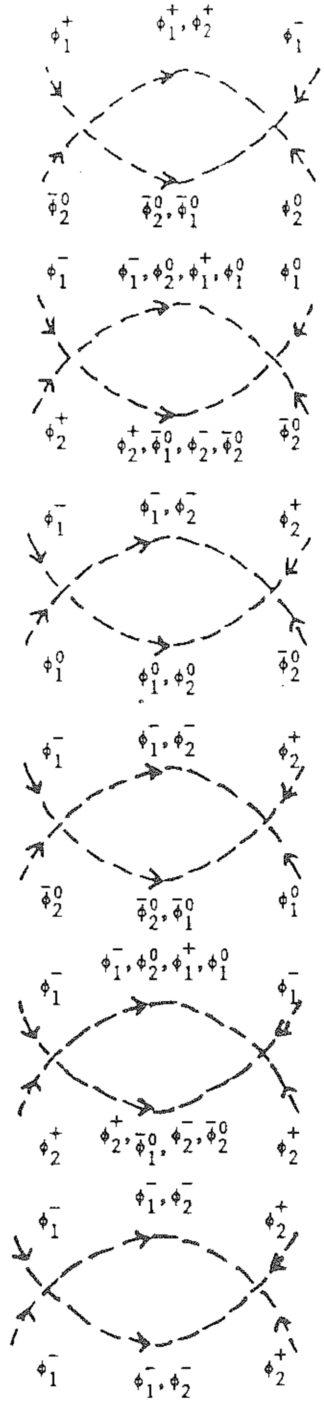
(c) + crossed $\quad 2((2\lambda_2)^2+\lambda_2^2+(\lambda_3+\lambda_4)^2+\lambda_3^2)$

(d) $\quad \frac{1}{2}(2\lambda_2)^2 + \frac{1}{2}(2\lambda_5)^2$

(e) $\quad 2\lambda_1\lambda_3 + \lambda_1(\lambda_3+\lambda_4) + \lambda_2(\lambda_3+\lambda_4) + 2\lambda_2\lambda_3$

(f) $\quad \lambda_3^2 + \lambda_4^2$

Fig.10. Higgs-boson contribution to the β functions for the vertices $2\lambda_1$ (a-b), $2\lambda_2$ (c-d) and λ_3 (e-f) in a two-doublet model.



(g) $\lambda_3^2 + \lambda_5^2$

(h) $\lambda_4(\lambda_3+\lambda_4).2 + (2\lambda_5)\lambda_5.2$

(i) $\lambda_1\lambda_4 + \lambda_2\lambda_4$

(j) $\lambda_3\lambda_4 + \lambda_3\lambda_4$

(k) + crossed $2(2\lambda_5(\lambda_3+\lambda_4).2 + \lambda_4\lambda_5.2)$

(l) $\frac{1}{2}(2\lambda_1)(2\lambda_5) + \frac{1}{2}(2\lambda_2)(2\lambda_5)$

Fig.11. Higgs-boson contribution to the β functions for the vertices λ_3 (g), λ_4 (h-j) and $2\lambda_5$ (k-l) in a two-doublet model.

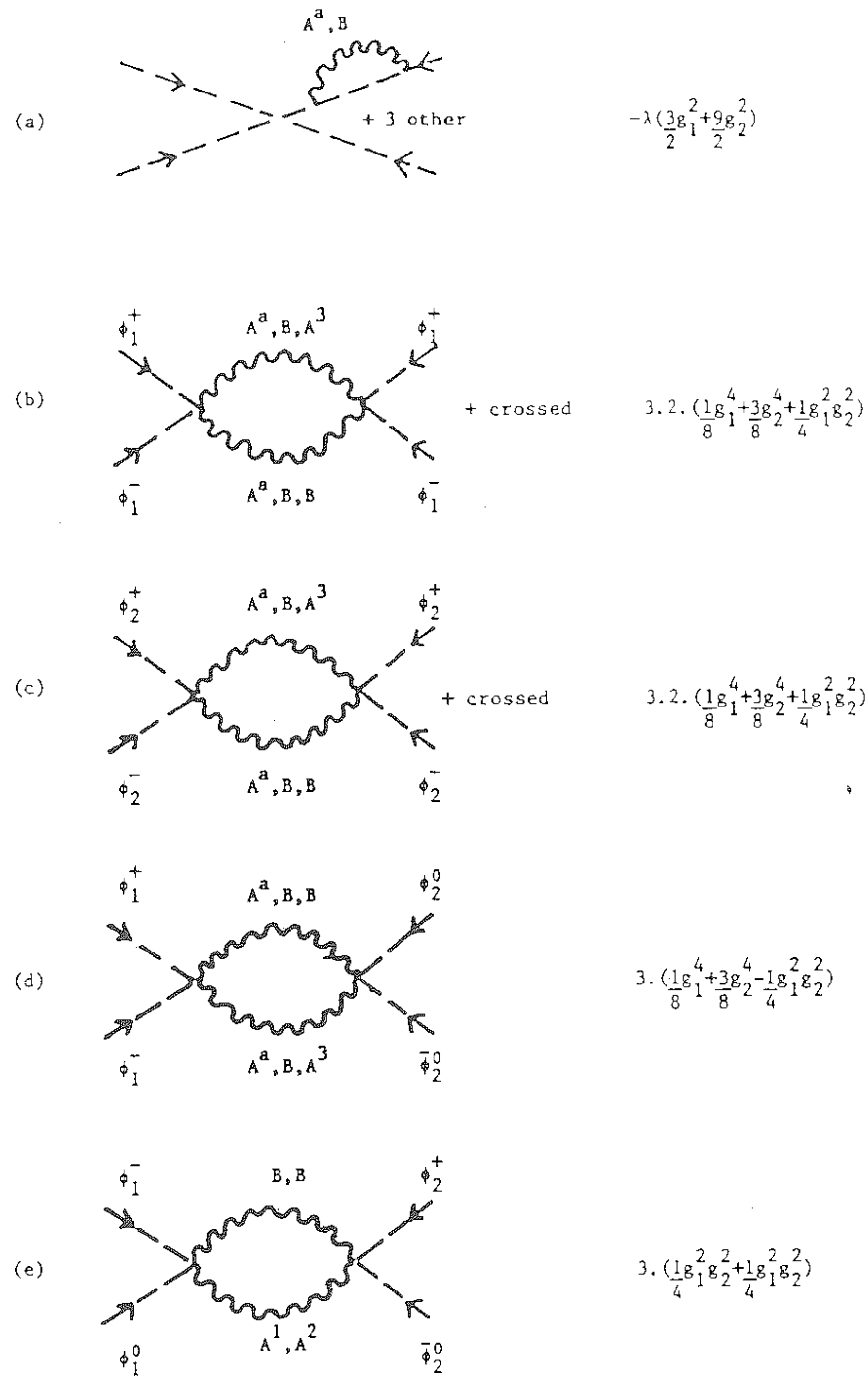


Fig.12. Gauge boson contribution to the β functions for $2\lambda_1$ (b), $2\lambda_2$ (c), λ_3 (d) and λ_4 (e). Diagram (a) contributes equally to all vertices including $2\lambda_5$.

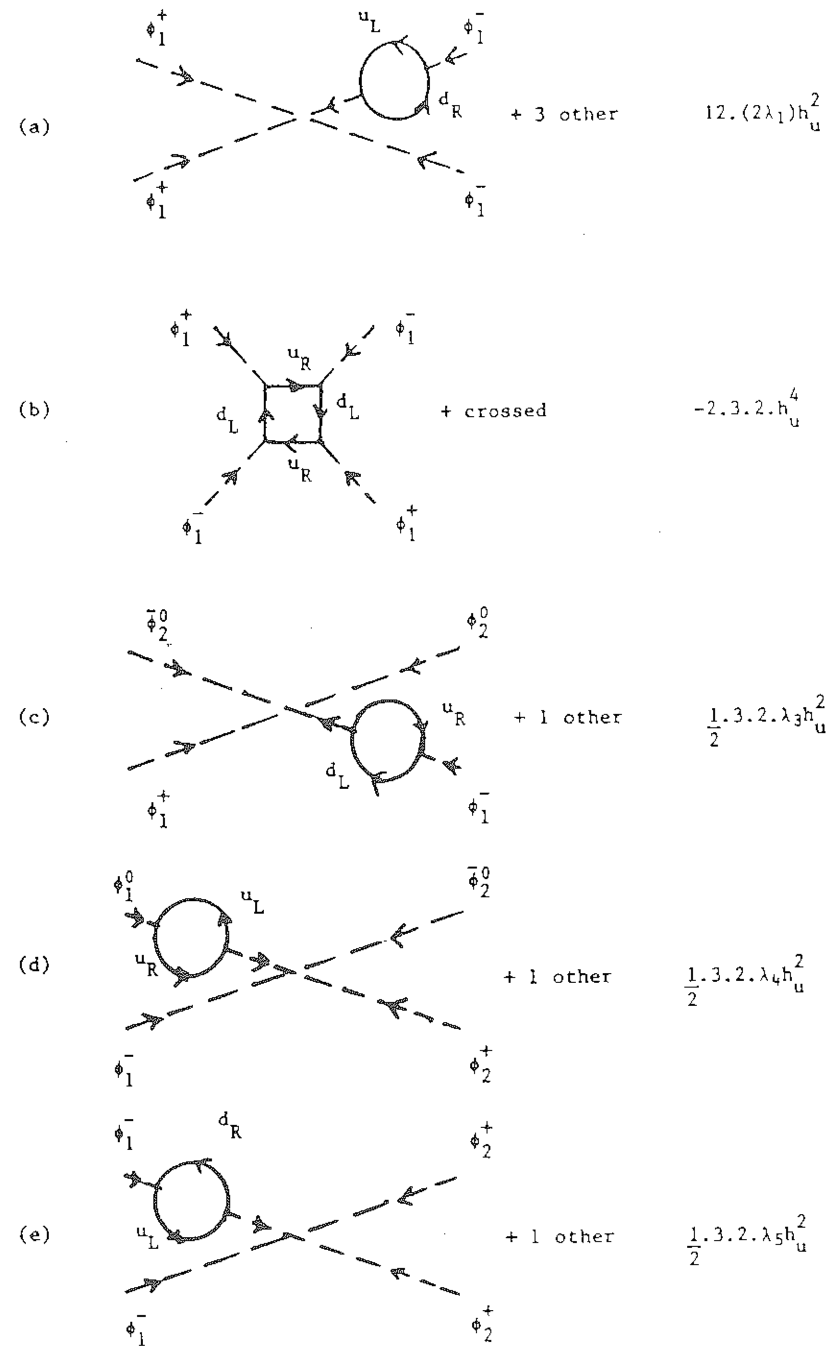


Fig.13. Up quark (which couples to ϕ_1) contribution to the β functions for $2\lambda_1$ (a-b), λ_3(c) λ_4 (d) and $2\lambda_5$ (e).