SDOF systems- a recap

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CIVE 405

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SDOF system

- A single geometrical co-ordinate often called a degree-of-freedom is adequate to represent the motion of the system

- A handy approximation for most of the complex engineering systems

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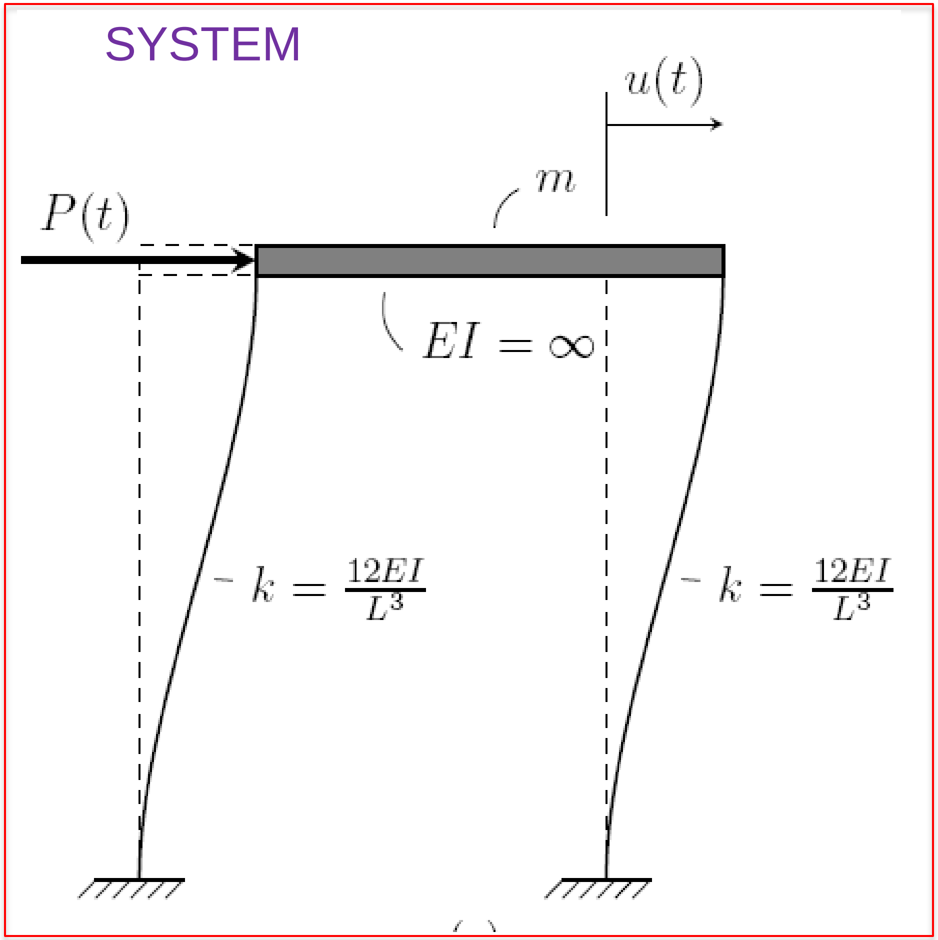
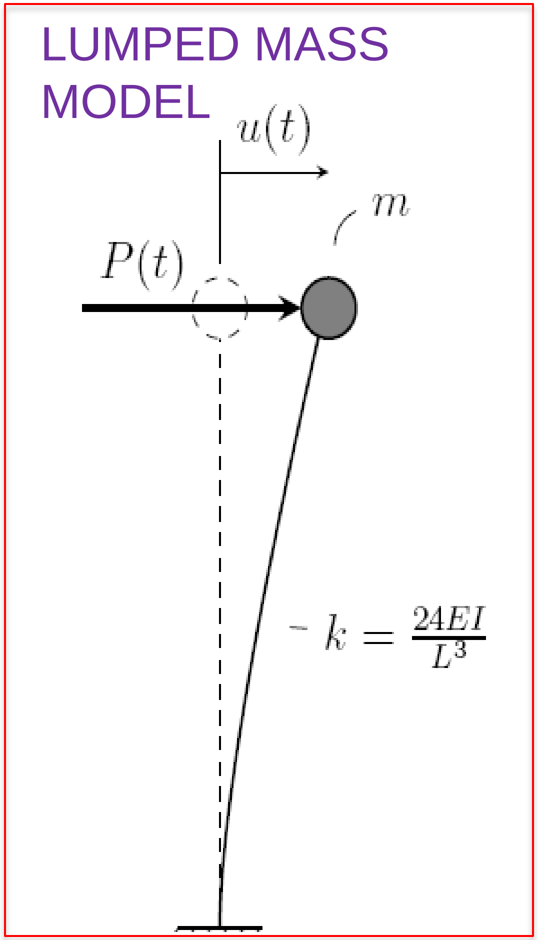
SDOF system: A physical model





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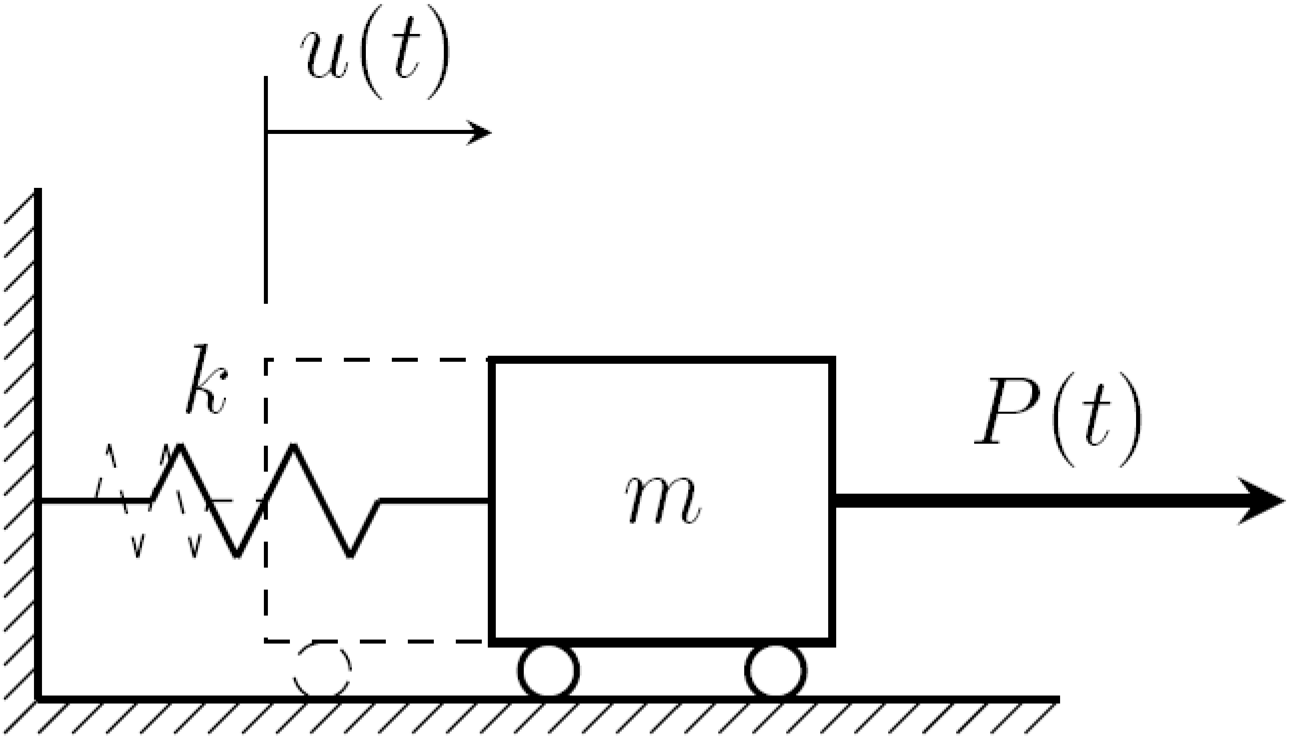
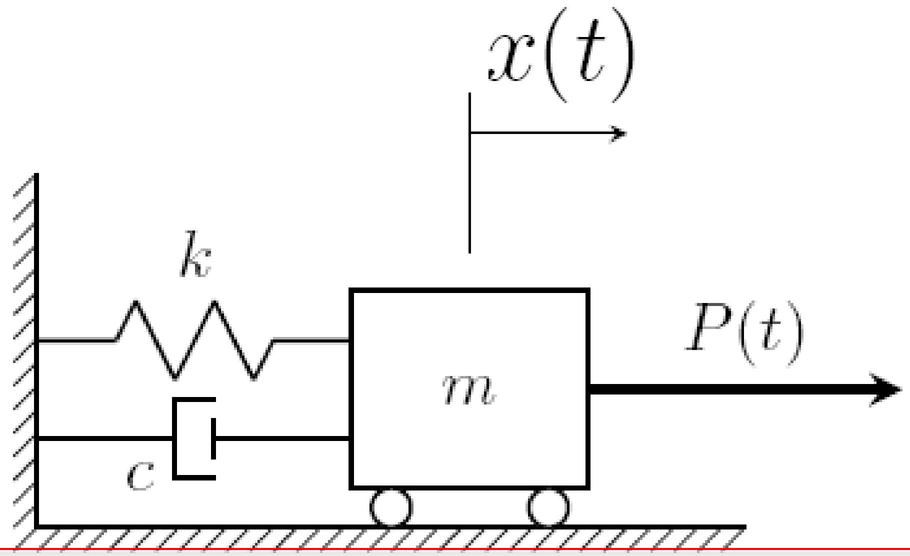
A mass-spring model



Spring-mass-damper model



- Inertia force $m\ddot{x}(t)$
- Spring force $c\dot{x}(t)$
- Damping force $kx(t)$

Equation of motion

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = P(t)$$

- Natural frequency $\omega_n = \sqrt{\frac{k}{m}}$
- Damping $\zeta = \frac{c}{2\sqrt{km}}$

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Writing equation of motion

- Force equilibrium
- Moment equilibrium
- Principal of virtual work
- Lagrange’s equations

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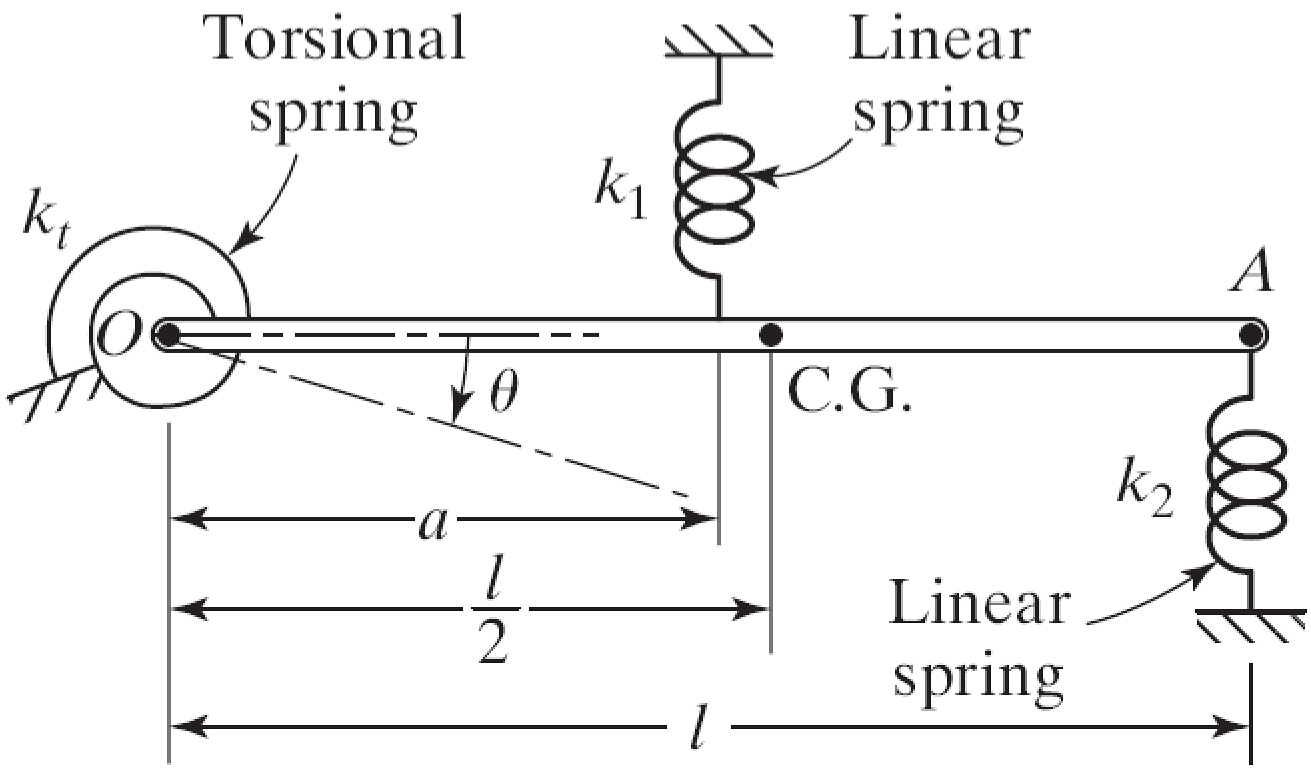
Example: Writing equation of motion



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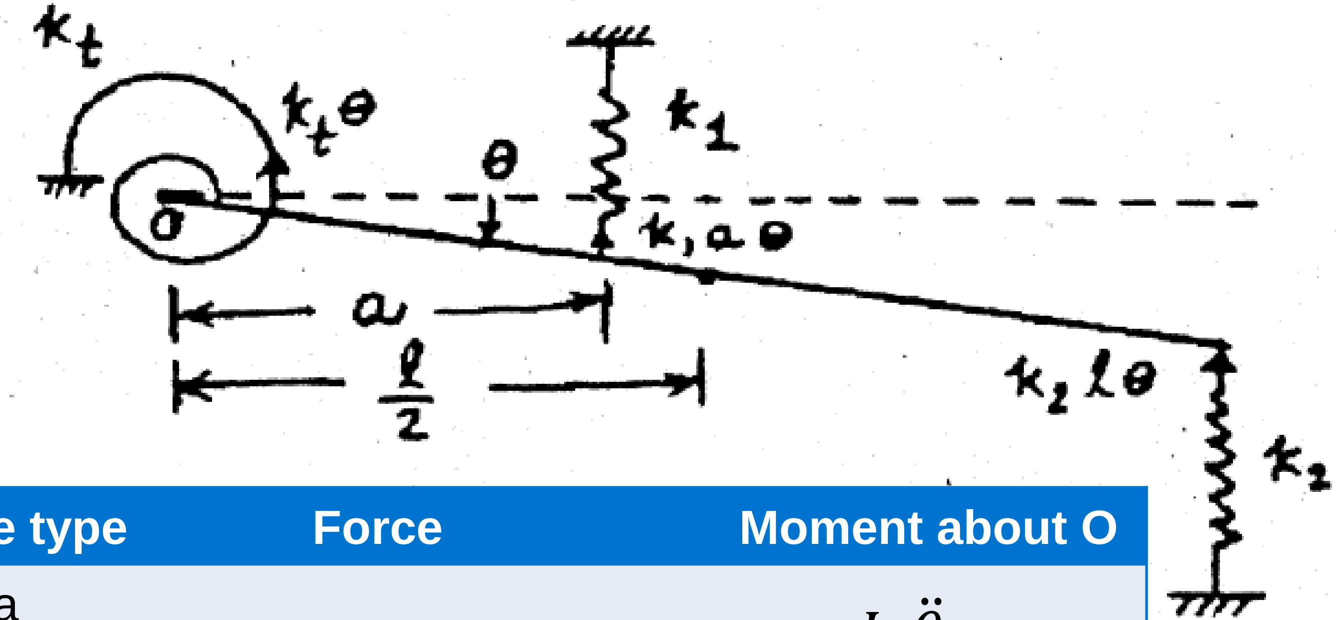
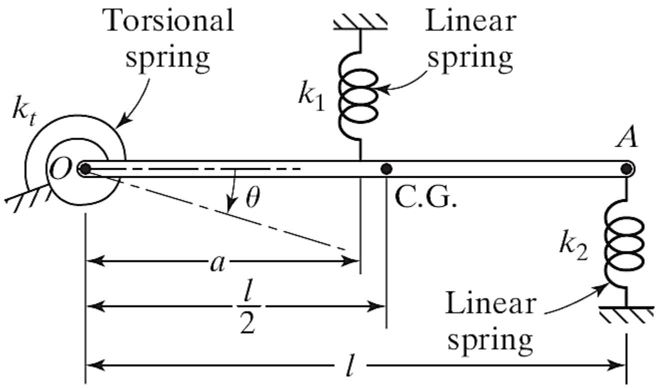
Example: Moment equilibrium



Force type	Force	Moment about O
Inertia		$J_o\ddot{\theta}$
Spring-1	$k_1 a\theta$	$(k_1 a\theta)a$
Spring-2	$k_2 l\theta$	$(k_2 l\theta)l$
Tor spring		$k_t\theta$

Example….



$$J_o = \frac{ml^2}{12} + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3}$$

Equation of motion ($\sum M = 0$ about the pt **O**)

$$\frac{ml^2}{3}\ddot{\theta} + (k_t + k_1 a^2 + k_2 l^2)\theta = 0$$

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Free vibration

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x = 0$$

=0

Assume a solution of the form: $x(t) = Ce^{\lambda t}$

For un-damped case the general solution is :

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t$$

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Free vibration…

- The constants A_1 & A_2 are found using initial conditions
- $$x(t = 0) = A_1 = x_0$$
  $$\dot{x}(t = 0) = \omega_n A_2 = \dot{x}_0$$

Complete solution: $$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

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Damped free vibration

For the **under-damped** case the general solution is :

$$x(t) = e^{-\zeta\omega_n t}\left(A\cos\omega_D t + B\sin\omega_D t\right)$$

Initial conditions: $x_0 = x(0)$

$$v_0 = \dot{x}(0)$$

Complete solution:

$$x(t) = e^{-\zeta\omega_n t}\left(x_0\cos\omega_D t + \frac{v_0 + \zeta\omega_n x_0}{\omega_D}\sin\omega_D t\right)$$

$$\omega_D = \omega_n\sqrt{1-\zeta^2}$$

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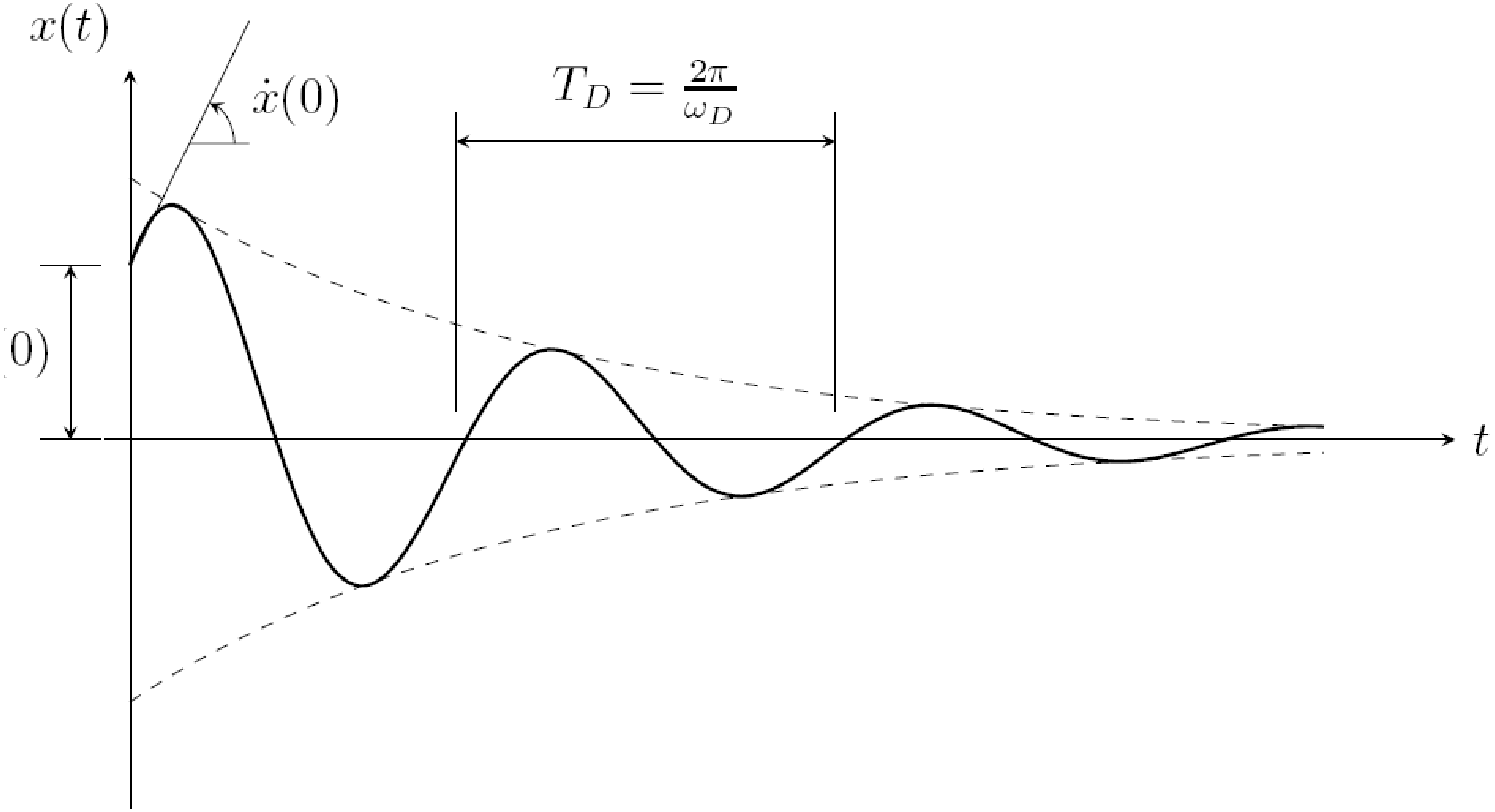
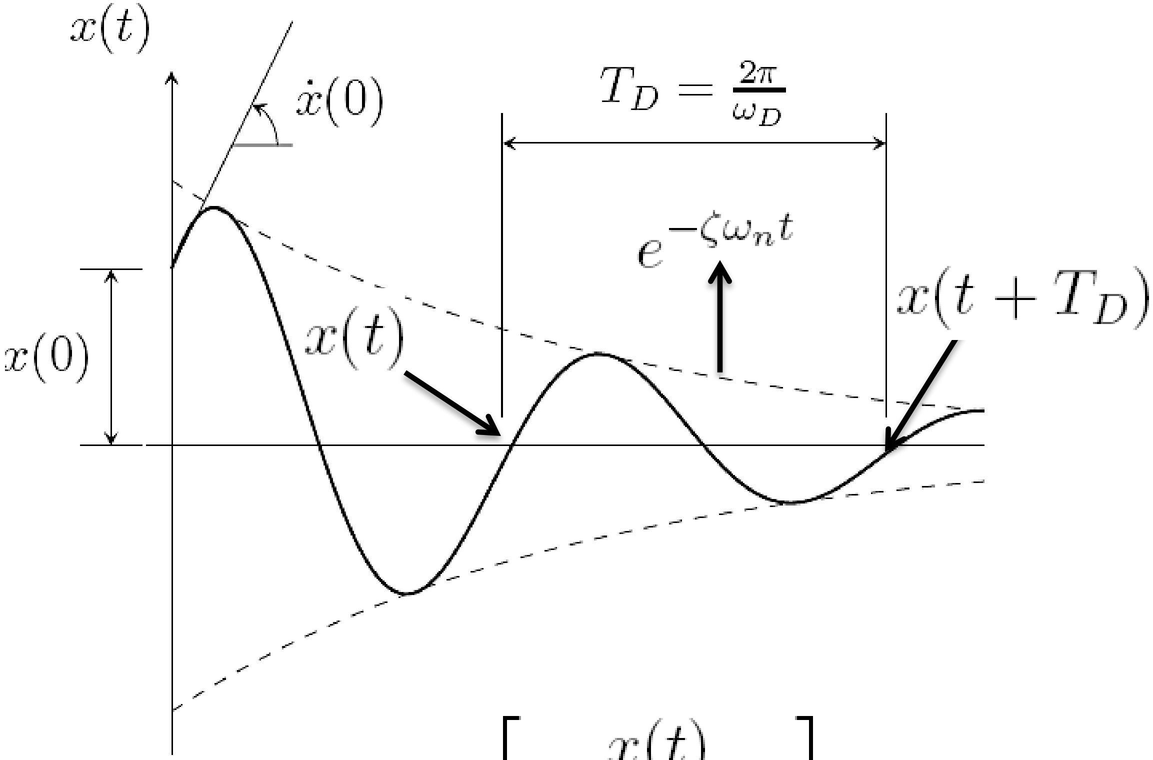
Logarithmic decrement



Logarithmic decrement



$$\ln\left[\frac{x(t)}{x(t+T_D)}\right] = 2\pi\zeta \qquad (\zeta < 0.3)$$

Reasonably accurate for most practical structures

Forced vibration under harmonic excitation

EQN of motion

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = P_0 \sin \omega t$$

$$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = \frac{P_0}{k}\omega_n^2 \sin \omega t$$

Assume the steady state solution :

$$x(t) = M \sin \omega t + N \cos \omega t$$

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harmonic excitation…..

$$\frac{x(t)}{x_{st}} = \frac{1-\phi^2}{\left(1-\phi^2\right)^2+4\zeta^2\phi^2}\sin\omega t + \frac{-2\zeta\phi}{\left(1-\phi^2\right)^2+4\zeta^2\phi^2}\cos\omega t$$

where $\phi = \frac{\omega}{\omega_n}$ and $x_{st} = \frac{P_0}{k}$

=> $$\frac{x(t)}{x_{st}} = U\sin\left(\omega t - \alpha\right)$$

$$U = AF = \frac{1}{\sqrt{\left(1-\phi^2\right)^2+4\zeta^2\phi^2}}$$

& $$\alpha = \tan^{-1}\left(\frac{2\zeta\phi}{1-\phi^2}\right)$$

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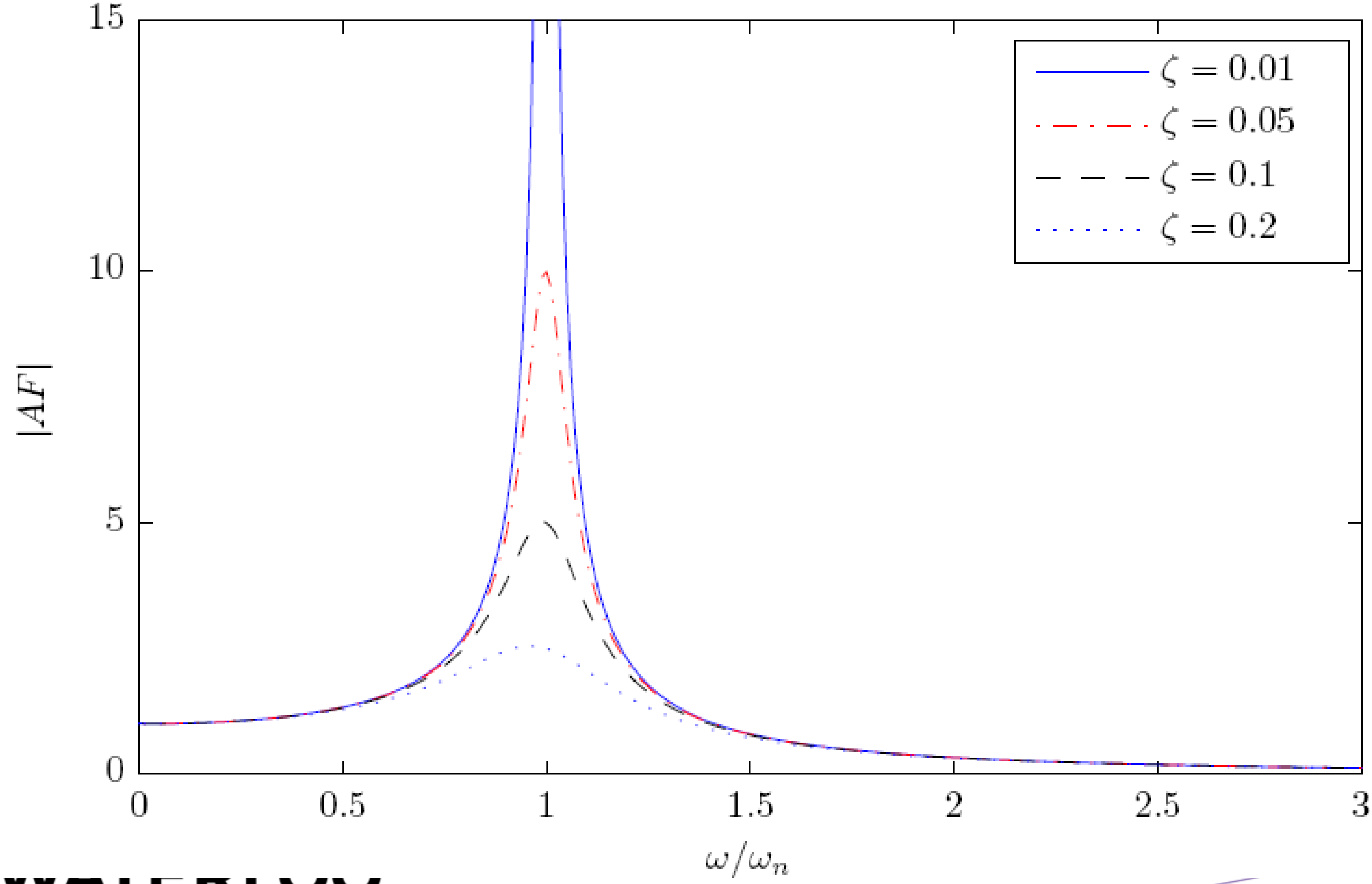
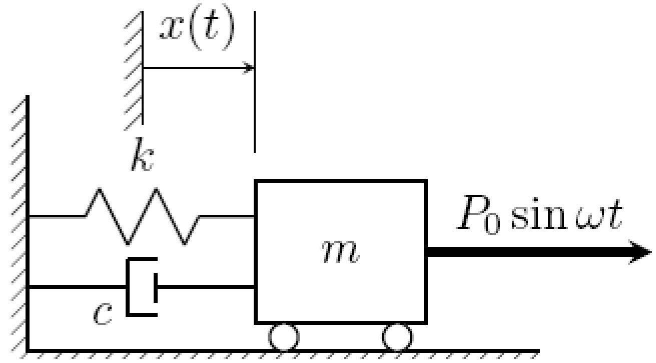
Amplification factor

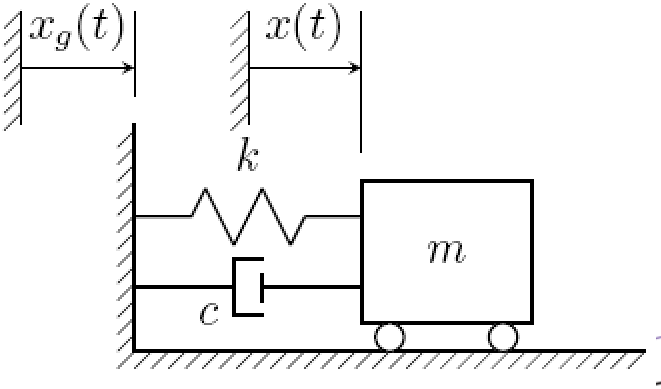


Applications of harmonic excitation

- Force transmissibility



- Base motion
  (base isolation)



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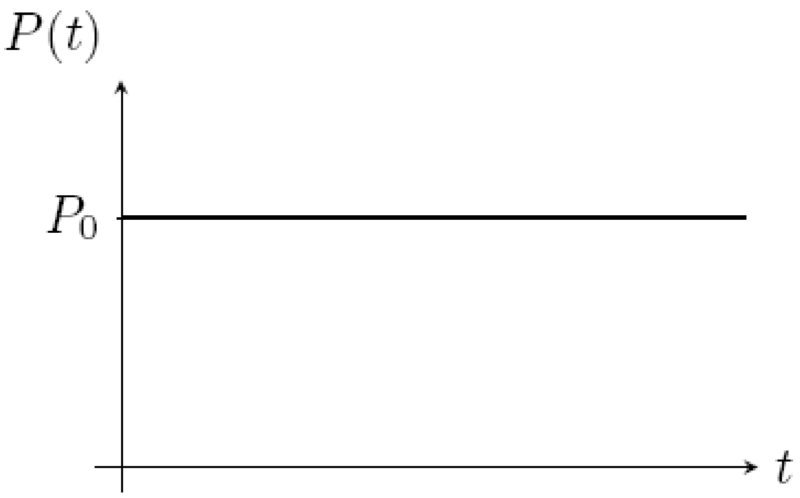
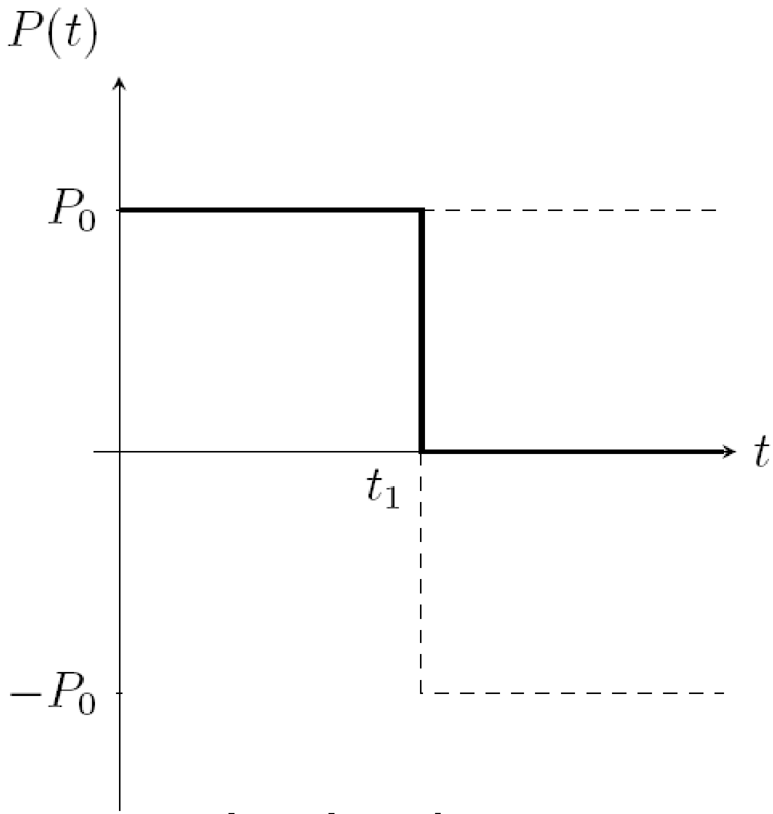
Forced vibration under arbitrary excitation



Step load or a constant load



Pulse load

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SDOF system under step input

SDOF system subjected to a step input

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = P_0$$

Initial conditions

$$x(0) = \dot{x}(0) = 0$$

$$x(t) = x_c(t) + x_p(t)$$

$$= e^{-\zeta\omega_n t}\left(A\cos\omega_D t + B\sin\omega_D t\right) + \frac{P_0}{k}$$

$$x(t) = x_{st}\left[1 - e^{-\omega_n\zeta t}\left(\cos\omega_D t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D t\right)\right]$$

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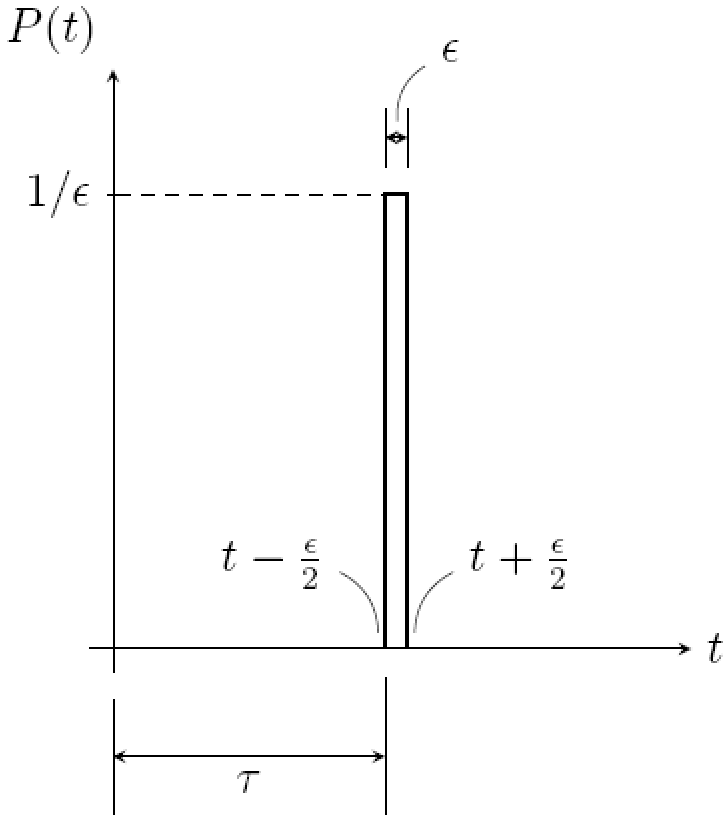
SDOF system under impulse



Dirac delta function

$$\delta(t-\tau) = \begin{cases} +\infty, & ; t = \tau \\ 0 & ; t \neq \tau \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)\ dt = 1 \quad = m\Delta\dot{x}(t=\tau)$$

$$\dot{x}(\tau) = \frac{1}{m}$$

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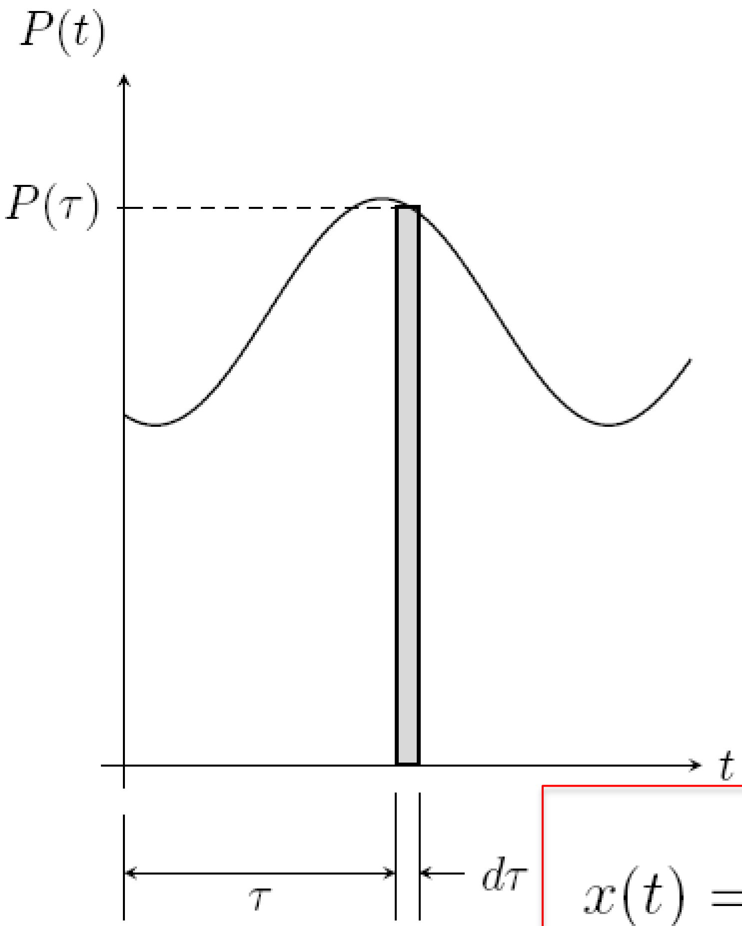
SDOF system under impulse

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \delta(t-\tau)$$

Solution:

$$x(t-\tau) = \frac{1}{m\omega_D} e^{-\zeta\omega_n(t-\tau)} \sin \omega_D (t-\tau)$$

Duhamel Integral



$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = P(t)$$

$$dx = \frac{P(\tau)d\tau}{m\omega_D} e^{-\zeta\omega_n(t-\tau)} \sin\omega_D (t-\tau)$$

$$x(t) = \frac{1}{m\omega_D} \int_0^t P(\tau) e^{-\zeta\omega_n(t-\tau)} \sin\omega_D (t-\tau) \ d\tau$$

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Frequency response Function

Approach the solution to the equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t).$$

using Fourier transforms

$$\left[(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2\right] X(\omega) = F(\omega)$$

Therefore,

$$X(\omega) = \frac{F(\omega)}{(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2}$$

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Frequency response Function

- The quantity $$\frac{X(\omega)}{F(\omega)} = H(i\omega) = \frac{1}{(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2}$$

  is called frequency response function

- Now consider the same equation of motion with

$$F(t) = \delta(t)$$

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Frequency & impulse response function

- $F(t) = \delta(t)$
- By def the response to a unit impulse load is called the impulse response function and is denoted as $g(t)$

The equation of motion is given by

$$\ddot{g} + 2\zeta\omega_n\dot{g} + \omega_n^2 g = \delta(t).$$

- Take the Fourier transform of the equation of motion,

$$G(\omega) = \frac{1}{2\pi\left[(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2\right]}$$

Frequency & impulse response function

- $$G(\omega) = \frac{1}{2\pi\left[(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2\right]}$$
- Then $$X(\omega) = 2\pi G(\omega) F(\omega)$$

and $G(\omega)$ is related to $H(\omega)$ by

$$H(\omega) = 2\pi G(\omega)$$

- $$H(\omega) = \int_{-\infty}^{\infty} g(t) \exp(-i\omega t)\, dt$$

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