CE 513: Statistical Methods in CE

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ITO CALCULUS

General Ito Integrals

$$I_n(X) = \sum_{i=1}^{n} X(s_{i-1})\ (W(s_i) - W(s_{i-1}))$$

• If X (t) *is a process with finite variance where the variance varies continuously in* with time, and

•If X (t) *only depends on the past of the WP, W (s) with s<=t,*

Then the Ito sum converges uniquely and independently of the partition.

$$\int_0^t X(s)\, dW(s)$$

General Ito Integrals

Proposition *Let $X(s)$ be a stochastic process on $[0,t]$ with two properties:*

(i) $\mu_2(s) = E\left(X^2(s)\right) < \infty$ *is a continuous function,*
(ii) $X(s)$ *is independent of* $W(s_j) - W(s_i)$ *with* $s \le s_i < s_j$.

Then it holds that

(a) the sum from converges in mean square:

$$\sum_{i=1}^{n} X(s_{i-1})\,(W(s_i) - W(s_{i-1})) \xrightarrow{2} \int_0^t X(s)\,dW(s)\,;$$

(b) the moments of the Ito integral are determined as:

$$E\left(\int_0^t X(s)\,dW(s)\right) = 0\,, \quad Var\left(\int_0^t X(s)\,dW(s)\right) = \int_0^t E\left(X^2(s)\right)\,ds\,.$$

Diffusions

• For modeling Stochastic-Differential Equations, Ito integral is an important ingredient.

• However, it gains its true importance only when combined with Riemann integrals.

• In general the sum of both integrals constitutes so-called **diffusions**

Hence, we now define processes $X(t)$ (with starting value $X(0)$)

$$X(t) = X(0) + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dW(s)\,.$$

Frequently, we will write this integral equation in differential form :

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t)\,.$$

Diffusions

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$$dX(t) = \mu(t)\, dt + \sigma(t)\, dW(t)\,.$$

In general, $\mu(s)$ and $\sigma(s)$ are stochastic; particularly, they are allowed to be dependent on $X(s)$ itself. Therefore, we write $\mu(s)$ and $\sigma(s)$ as abbreviations for functions which firstly explicitly depend on time and secondly depend on X simultaneously:

$$\mu(s) = \mu\,(s, X(s))\,, \quad \sigma(s) = \sigma\,(s, X(s))$$

Processes μ and σ satisfying these conditions are used to define diffusions $X(t)$:

$$dX(t) = \mu\,(t, X(t))\; dt + \sigma\,(t, X(t))\; dW(t)\,, \quad t \in [0, T]\,.$$

Diffusions

Example *(Brownian Motion with Drift)* We consider the Brownian motion with drift and a starting value 0:

$$X(t) = \mu t + \sigma W(t)$$
$$= \mu \int_0^t ds + \sigma \int_0^t dW(s) .$$

Therefore, the differential notation reads

$$dX(t) = \mu\, dt + \sigma\, dW(t) .$$

Hence, this is a diffusion whose drift and volatility are constant:

$$\mu\,(t, X(t)) = \mu \quad \text{and} \quad \sigma\,(t, X(t)) = \sigma .$$

Propositions

Proposition **(Symbolic Notation)** *It holds for* $n \to \infty$

$$(a) \sum_{i=1}^{n} (W(s_i) - W(s_{i-1}))^2 \xrightarrow{2} \int_0^t (dW(s))^2 = t\,,$$

$$(b) \sum_{i=1}^{n} (W(s_i) - W(s_{i-1}))\ (s_i - s_{i-1}) \xrightarrow{2} \int_0^t dW(s)\, ds = 0$$

$$(c) \sum_{i=1}^{n} (s_i - s_{i-1})^2 \to \int_0^t (ds)^2 = 0\,.$$

Example

Prove: $$\sum_{i=1}^{n} (W(s_i) - W(s_{i-1}))\ (s_i - s_{i-1}) \xrightarrow{2} \int_0^t dW(s)\, ds = 0$$

Let us call the aforementioned covariation CV_n:

$$CV_n = \sum_{i=1}^{n} (W(s_i) - W(s_{i-1}))\ (s_i - s_{i-1})\,.$$

The claim reads: $\mathrm{MSE}(CV_n, 0) \to 0$. As it obviously holds that $\mathrm{E}(CV_n) = 0$, we obtain

$$\mathrm{MSE}(CV_n, 0) = \mathrm{Var}(CV_n)\,.$$

Hence, it remains to be shown that this variance tends to zero: Due to the independence of the increments of the WP, one determines

$$\mathrm{Var}(CV_n) = \sum_{i=1}^{n} \mathrm{Var}\,(W(s_i) - W(s_{i-1}))\ (s_i - s_{i-1})^2\,,$$

Example

and hence

$$
\begin{aligned}
\mathrm{Var}(CV_n) &= \sum_{i=1}^{n} (s_i - s_{i-1})^3 \\
&\leq \max_{1 \leq i \leq n} (s_i - s_{i-1}) \sum_{i=1}^{n} (s_i - s_{i-1})^2 \\
&= \max_{1 \leq i \leq n} (s_i - s_{i-1})\, Q_n(id, t) \\
&\to 0\,,
\end{aligned}
$$

ITO's Lemma

Let us recall the Ito-Integral for WP which can be written equivalently as

$$2\int_0^t W(s)\,dW(s) = W^2(t) - t\,.$$

If $g(W) = W^2$ is defined with derivatives $g'(W) = 2W$ and $g''(W) = 2$, then this equation can also be formulated as follows:

$$\int_0^t g'\,(W(s))\;dW(s) = g\,(W(t)) - t$$

$$= g\,(W(t)) - \frac{1}{2}\int_0^t g''\,(W(s))\;ds\,.$$

ITO's Lemma

Corollary 1 **(Ito's Lemma for WP)** *Let $g : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Then it holds that*

$$dg\,(W(t)) = g'\,(W(t))\;dW(t) + \frac{1}{2}\,g''\,(W(t))\;dt\,.$$

In integral form this corollary to Ito's lemma is to be read as follows:

$$g\,(W(t)) = g\,(W(0)) + \int_0^t g'\,(W(s))\;dW(s) + \frac{1}{2}\int_0^t g''\,(W(s))\;ds\,.$$

Strictly speaking, this integral equation is the statement of the corollary, which is abbreviated by the differential notation. However, in doing so it must not be forgotten that the WP is not differentiable. Sometimes one also writes even more briefly:

$$dg\,(W) = g'(W)\,dW + \frac{1}{2}\,g''(W)\,dt\,.$$

ITO's Lemma for Diffusions

Proposition 1 **(Ito's Lemma with One Dependent Variable)** *Let* $g : \mathbb{R} \to \mathbb{R}$ *be twice continuously differentiable and* $X(t)$ *a diffusion on* $[0, T]$

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dW(t)\,.$$

Then it holds that

$$dg\,(X(t)) = g'\,(X(t))\; dX(t) + \frac{1}{2}\, g''\,(X(t))\; \sigma^2(t)\, dt\,.$$

If $X(t) = W(t)$ is a Wiener process, i.e. $\mu(t) = 0$ and $\sigma(t) = 1$, then Corollary 1 is obtained as a special case.

The statement in Proposition 1 is given somewhat succinctly It can be condensed even more by suppressing the dependence on time:

$$dg\,(X) = g'\,(X)\; dX + \frac{1}{2}\, g''\,(X)\; \sigma^2\, dt\,.$$

Example

Example (Differential of the Exponential Function) Let a diffusion $X(t)$ be given,

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t)\,.$$

Then, how does the differential of $e^{X(t)}$ read? This example is particularly easy to calculate as it holds for $g(x) = e^x$ that:

$$g''(x) = g'(x) = g(x) = e^x.$$

Hence Ito's lemma yields:

$$\begin{aligned} de^{X(t)} &= e^{X(t)}dX(t) + \frac{e^{X(t)}}{2}\sigma^2(t)dt \\ &= e^{X(t)}\left(\mu(t) + \frac{\sigma^2(t)}{2}\right)dt + e^{X(t)}\,\sigma(t)dW(t). \end{aligned}$$

If $X(t)$ is deterministic, i.e. $\sigma(t) = 0$, then it results

$$\frac{de^{X(t)}}{dt} = e^{X(t)}\frac{dX(t)}{dt},$$

which just corresponds to the traditional chain rule (outer derivative times inner derivative).

Bivariate Diffusions: one factor case

Let the function *g be dependent on two diffusions* X_1 *and* X_2*, where both are driven by the very same Wiener process. Occasionally, we* call this case as the one-factor case as it is the identical factor *W (t) is driving both the diffusions.*

Let g be a function in two arguments, whose partial derivatives are denoted by

$$\frac{\partial g(X_1, X_2)}{\partial X_i} \quad \text{and} \quad \frac{\partial^2 g(X_1, X_2)}{\partial X_i \, \partial X_j} .$$

Then, the following proposition is a special case of Proposition

(Ito's Lemma with Two Dependent Variables) *Let* $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *be twice continuously differentiable with respect to both arguments, and let* $X_i(t)$ *be diffusions on* $[0, T]$ *with the same WP:*

$$dX_i(t) = \mu_i(t)\, dt + \sigma_i(t)\, dW(t) , \quad i = 1, 2 .$$

Bivariate Diffusions: one factor case

Proposition. 2

Then it holds that

$$dg\,(X_1(t),\,X_2(t)) = \frac{\partial g\,(X_1(t),\,X_2(t))}{\partial X_1}\,dX_1(t) + \frac{\partial g\,(X_1(t),\,X_2(t))}{\partial X_2}\,dX_2(t)$$
$$+ \frac{1}{2}\left[\frac{\partial^2 g\,(X_1(t),\,X_2(t))}{\partial X_1^2}\,\sigma_1^2(t) + \frac{\partial^2 g\,(X_1(t),\,X_2(t))}{\partial X_2^2}\,\sigma_2^2(t)\right]dt$$
$$+ \frac{\partial^2 g\,(X_1(t),\,X_2(t))}{\partial X_1\,\partial X_2}\,\sigma_1(t)\,\sigma_2(t)\,dt\,.$$

Note that substitution of $dX_i(t)$ in Proposition leads again to an integral equation for the process $g\,(X_1(t),\,X_2(t))$ including Riemann and Ito integrals.

Bivar-diffusions: 1-factor invariant case

Frequently, the time dependence of the processes will be suppressed in order to obtain a more economical formulation of Proposition :

$$dg(X_1, X_2) = \frac{\partial g(X_1, X_2)}{\partial X_1} dX_1 + \frac{\partial g(X_1, X_2)}{\partial X_2} dX_2 + \frac{1}{2}\left[\frac{\partial^2 g(X_1, X_2)}{\partial X_1^2}\sigma_1^2 + \frac{\partial^2 g(X_1, X_2)}{\partial X_2^2}\sigma_2^2\right] dt + \frac{\partial^2 g(X_1, X_2)}{\partial X_1 \partial X_2}\sigma_1 \sigma_2 dt .$$

Exercise

Prove that the previous proposition is equivalent to following statement

$$
\begin{aligned}
dg(X_1, X_2) = & \frac{\partial g(X_1, X_2)}{\partial X_1} dX_1 + \frac{\partial g(X_1, X_2)}{\partial X_2} dX_2 \\
& + \frac{1}{2}\left[\frac{\partial^2 g(X_1, X_2)}{\partial X_1^2}(dX_1)^2 + \frac{\partial^2 g(X_1, X_2)}{\partial X_2^2}(dX_2)^2\right] \\
& + \frac{1}{2}\left[\frac{\partial^2 g(X_1, X_2)}{\partial X_1 \partial X_2} dX_1 dX_2 + \frac{\partial^2 g(X_1, X_2)}{\partial X_2 \partial X_1} dX_2 dX_1\right]
\end{aligned}
$$

Example: One factor product rule

$$d(X_1(t)\,X_2(t)) = X_2(t)\,dX_1(t) + X_1(t)\,dX_2(t) + \sigma_1(t)\,\sigma_2(t)\,dt\,.$$

Observe that under $\sigma_1(t) = 0$ or $\sigma_2(t) = 0$ (no stochastics)

well-known product rule is just reproduced

The derivation of follows for $g(x_1, x_2) = x_1\,x_2$ with

$$\frac{\partial g}{\partial x_1} = x_2, \quad \frac{\partial^2 g}{\partial x_1^2} = 0, \qquad \text{and} \qquad \frac{\partial^2 g}{\partial x_1\,\partial x_2} = \frac{\partial^2 g}{\partial x_2\,\partial x_1} = 1$$

$$\frac{\partial g}{\partial x_2} = x_1, \quad \frac{\partial^2 g}{\partial x_2^2} = 0$$

Hence, we obtain an abbreviated form:

$$d\,(X_1\,X_2) = \frac{\partial g\,(X_1,\,X_2)}{\partial X_1}\,dX_1 + \frac{\partial g\,(X_1,\,X_2)}{\partial X_2}\,dX_2 + \sigma_1\,\sigma_2\,dt\,,$$

where the second derivatives were plugged in. If one substitutes the first derivatives, then one obtains the result

Ito's Lemma with Time as a Dependent Variable

(Ito's Lemma with Time as a Dependent Variable) *Let* $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ *be twice continuously differentiable with respect to both arguments and let* $X(t)$ *be a diffusion on* $[0, T]$

Then it holds that

$$dg\,(t,\,X(t)) = \frac{\partial g\,(t,\,X(t))}{\partial t}\,dt + \frac{\partial g\,(t,\,X(t))}{\partial X}\,dX(t) + \frac{1}{2}\,\frac{\partial^2 g\,(t,\,X(t))}{\partial X^2}\,\sigma^2(t)\,dt$$

Again, suppressing time-dependence, this can be condensed to

$$dg\,(t,\,X) = \frac{\partial g\,(t,\,X)}{\partial t}\,dt + \frac{\partial g\,(t,\,X)}{\partial X}\,dX + \frac{1}{2}\,\frac{\partial^2 g\,(t,\,X)}{\partial X^2}\,\sigma^2\,dt\,.$$

K-Variate Diffusions

Let us consider as a generalization the case where g depends on K diffusions, all of them given by the identical WP:

$$g\colon \mathbb{R}^K \to \mathbb{R}\,, \quad \text{i.e.} \quad g = g\,(X_1, \ldots, X_K) \in \mathbb{R}\,.$$

Then it holds with $dX_k(t)$, $k = 1, \ldots, K$, due to a second order Taylor expansion that:

$$dg\,(X_1, \ldots, X_K) = \sum_{k=1}^{K} \frac{\partial g}{\partial X_k}\, dX_k + \frac{1}{2} \sum_{k=1}^{K} \sum_{j=1}^{K} \frac{\partial^2 g}{\partial X_k\, \partial X_j}\, dX_k\, dX_j$$

As in the bivariate case, one obtains $dX_k\, dX_j = \sigma_k\, \sigma_j\, dt$. Sometimes, time as a further variable is allowed for,

$$g\colon [0, T] \times \mathbb{R}^K \to \mathbb{R}\,, \quad \text{i.e.} \quad g = g\,(t, X_1, \ldots, X_K) \in \mathbb{R}\,,$$

and

$$dg\,(t, X_1, \ldots, X_K) = \frac{\partial g}{\partial t}\, dt + \sum_{k=1}^{K} \frac{\partial g}{\partial X_k}\, dX_k + \frac{1}{2} \sum_{k=1}^{K} \sum_{j=1}^{K} \frac{\partial^2 g}{\partial X_k\, \partial X_j}\, dX_k\, dX_j$$

The General Case

Now, $W_1(t), \ldots, W_d(t)$ denote stochastically independent standard Wiener processes. We allow for d factors driving each of the K diffusions. According to this, let $\mathbf{X}(t)$ be a K-dimensional diffusion[1] $\mathbf{X}'(t) = (X_1(t), \ldots, X_K(t))$, defined by d factors $W_j(t), j = 1, \ldots, d$:

$$dX_k(t) = \mu_k(t)dt + \sum_{j=1}^{d} \sigma_{kj}(t)dW_j(t), \quad k = 1, \ldots, K.$$

In order to have a diffusion, it holds for μ_k and σ_{kj} that they may only depend on $\mathbf{X}(t)$ and t:

$$\mu_k(t) = \mu_k(t, \mathbf{X}(t)), \quad k = 1, \ldots K,$$

$$\sigma_{kj}(t) = \sigma_{kj}(t, \mathbf{X}(t)), \quad k = 1, \ldots K, \; j = 1, \ldots d.$$

Ito's Lemma (Independent WP)): General Case

Proposition 3 **(Ito's Lemma (Independent WP))** *Let $g : \mathbb{R}^K \to \mathbb{R}$ be twice continuously differentiable with respect to all the arguments, and let $X_k(t)$ be diffusions on $[0, T]$ depending on d independent Wiener processes:*

$$dX_k(t) = \mu_k(t)dt + \sum_{j=1}^{d} \sigma_{kj}(t)dW_j(t), \quad k = 1, \ldots, K.$$

Then it holds for $\mathbf{X}'(t) = (X_1(t), \ldots, X_K(t))$ that

$$dg(\mathbf{X}(t)) = \sum_{k=1}^{K} \frac{\partial g(\mathbf{X}(t))}{\partial X_k} dX_k(t) + \frac{1}{2} \sum_{i=1}^{K} \sum_{k=1}^{K} \frac{\partial^2 g(\mathbf{X}(t))}{\partial X_i \partial X_k} dX_i(t)dX_k(t)$$

with

$$dX_i(t)dX_k(t) = \sum_{j=1}^{d} \sigma_{ij}(t)\sigma_{kj}(t)dt.$$

Example: 2-factor case

Let us consider the case $K = d = 2$. Then,

$$dg(\mathbf{X}(t)) = \sum_{k=1}^{2} \frac{\partial g(\mathbf{X}(t))}{\partial X_k} dX_k(t) + \frac{1}{2} \sum_{i=1}^{2} \sum_{k=1}^{2} \frac{\partial^2 g(\mathbf{X}(t))}{\partial X_i \partial X_k} dX_i(t) dX_k(t)$$

with

$$dX_1 dX_1 = (\sigma_{11}^2 + \sigma_{12}^2) dt\,,$$

$$dX_2 dX_2 = (\sigma_{21}^2 + \sigma_{22}^2) dt\,,$$

and

$$dX_1 dX_2 = (\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}) dt.$$

Two interesting special cases result:

1. $\sigma_{12} = \sigma_{22} = 0$ (one-factor model),
2. $\sigma_{12} = \sigma_{21} = 0$ (independent diffusions).

Example

(2-Factor Product Rule) Proposition 3 with $K = d = 2$ yields

$$d(X_1 X_2) = X_2\, dX_1 + X_1\, dX_2 + dX_1\, dX_2$$

In the second borderline case of stochastically independent diffusions, it holds, as in the deterministic case, that

$$d(X_1 X_2) = X_2\, dX_1 + X_1\, dX_2\,.$$

Without restrictions : $d(X_1 X_2) = X_2\, dX_1 + X_1\, dX_2 + (\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22})dt\,.$

The example illustrates that a proper application of Ito's lemma needs to account for the number of factors underlying the diffusions.

Stochastic Differential Equations: Intro

- At first, we have an ordinary differential equation (ODE):

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, t).$$

- Then we add white noise to the right hand side:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, t) + \mathbf{w}(t).$$

- Generalize a bit by adding a multiplier matrix on the right:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t)\,\mathbf{w}(t).$$

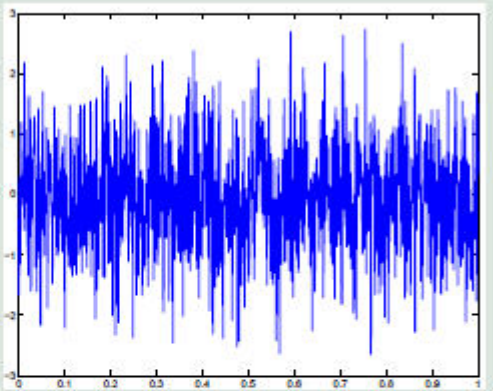
- Now we have a stochastic differential equation (SDE).
- $\mathbf{f}(\mathbf{x}, t)$ is the drift function and $\mathbf{L}(\mathbf{x}, t)$ is the dispersion matrix.

Stochastic Differential Equations: Intro

White noise

1. $\mathbf{w}(t_1)$ and $\mathbf{w}(t_2)$ are independent if $t_1 \neq t_2$.
2. $t \mapsto \mathbf{w}(t)$ is a Gaussian process with the mean and covariance:

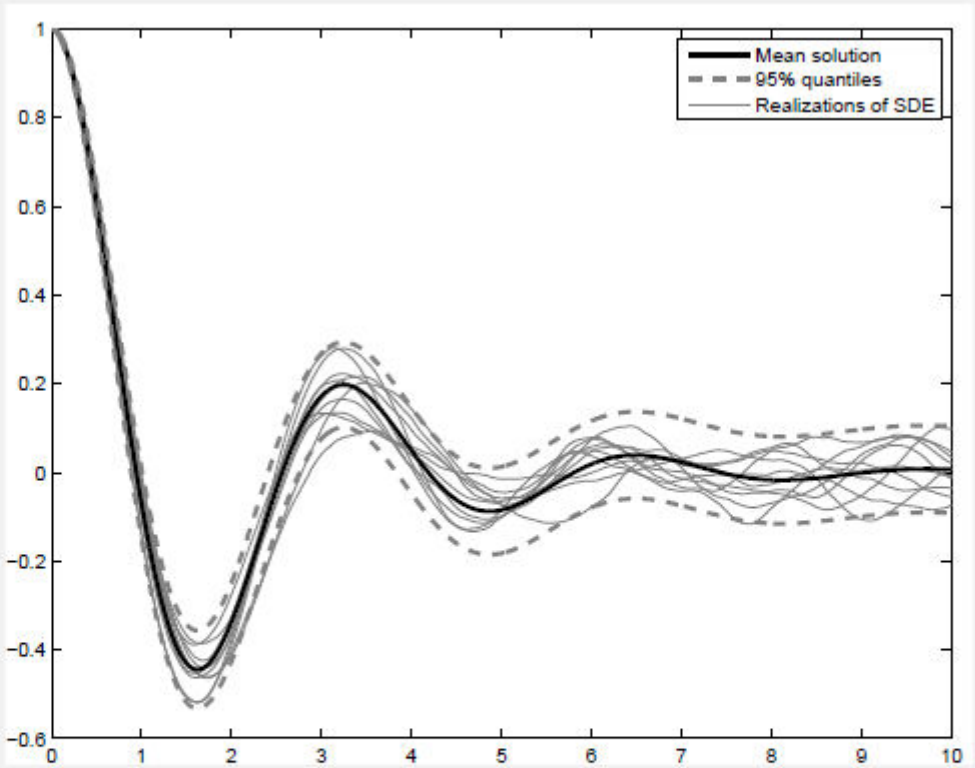
$$\mathrm{E}[\mathbf{w}(t)] = \mathbf{0}$$
$$\mathrm{E}[\mathbf{w}(t)\,\mathbf{w}^{\mathsf{T}}(s)] = \delta(t-s)\,\mathbf{Q}.$$



- $\mathbf{Q}$ is the spectral density of the process.
- The sample path $t \mapsto \mathbf{w}(t)$ is discontinuous almost everywhere.
- White noise is unbounded and it takes arbitrarily large positive and negative values at any finite interval.

Stochastic Differential Equations: Intro

What does a solution of SDE look like?



Paths of stochastic spring model

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \nu^2 x(t) = w(t).$$

Attempts to Solution

- Linear time-invariant stochastic differential equation (LTI SDE):

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{L}\,\mathbf{w}(t), \qquad \mathbf{x}(t_0) \sim \mathrm{N}(\mathbf{m}_0, \mathbf{P}_0).$$

- We can now take a "leap of faith" and solve this as if it was a deterministic ODE:

1. Move $\mathbf{F}\,\mathbf{x}(t)$ to left and multiply by integrating factor $\exp(-\mathbf{F}\,t)$:

$$\exp(-\mathbf{F}\,t)\,\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} - \exp(-\mathbf{F}\,t)\,\mathbf{F}\,\mathbf{x}(t) = \exp(-\mathbf{F}\,t)\,\mathbf{L}\,\mathbf{w}(t).$$

2. Rewrite this as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(-\mathbf{F}\,t)\,\mathbf{x}(t)\right] = \exp(-\mathbf{F}\,t)\,\mathbf{L}\,\mathbf{w}(t).$$

3. Integrate from t_0 to t:

$$\exp(-\mathbf{F}\,t)\,\mathbf{x}(t) - \exp(-\mathbf{F}\,t_0)\,\mathbf{x}(t_0) = \int_{t_0}^{t} \exp(-\mathbf{F}\,\tau)\,\mathbf{L}\,\mathbf{w}(\tau)\,\mathrm{d}\tau.$$

Attempts to Solution

- Rearranging then gives the solution:

$$\mathbf{x}(t) = \exp(\mathbf{F}\,(t - t_0))\,\mathbf{x}(t_0) + \int_{t_0}^{t} \exp(\mathbf{F}\,(t - \tau))\,\mathbf{L}\,\mathbf{w}(\tau)\,\mathrm{d}\tau.$$

- We have assumed that $\mathbf{w}(t)$ is an ordinary function, which it is not.
- Here we are lucky, because for linear SDEs we get the right solution, but generally not.
- The source of the problem is the integral of a non-integrable function on the right hand side.

Numerical approaches

- We could now attempt to analyze non-linear SDEs of the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t)\,\mathbf{w}(t)$$

- We cannot solve the deterministic case—no possibility for a "leap of faith".
- We don't know how to derive the mean and covariance equations.
- What we can do is to simulate by using Euler–Maruyama:

$$\hat{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \mathbf{f}(\hat{\mathbf{x}}(t_k), t_k)\,\Delta t + \mathbf{L}(\hat{\mathbf{x}}(t_k), t_k)\,\Delta\beta_k,$$

  where $\Delta\beta_k$ is a Gaussian random variable with distribution $\mathrm{N}(\mathbf{0}, \mathbf{Q}\,\Delta t)$.
- Note that the variance is proportional to Δt, not the standard derivation.

Numerical approaches

Equivalent integral equation

- Integrating the differential equation from t_0 to t gives:

$$\mathbf{x}(t) - \mathbf{x}(t_0) = \int_{t_0}^{t} \mathbf{f}(\mathbf{x}(t), t)\ \mathrm{d}t + \int_{t_0}^{t} \mathbf{L}(\mathbf{x}(t), t)\, \mathbf{w}(t)\ \mathrm{d}t.$$

- The first integral is just a normal Riemann/Lebesgue integral.
- The second integral is the problematic one due to the white noise.
- This integral cannot be defined as Riemann, Stieltjes or Lebesgue integral as we shall see next.

Numerical approaches

Attempt 1: Riemann integral

- In the Riemannian sense the integral would be defined as

$$\int_{t_0}^{t} \mathbf{L}(\mathbf{x}(t), t)\, \mathbf{w}(t)\, \mathrm{d}t = \lim_{n\to\infty} \sum_{k} \mathbf{L}(\mathbf{x}(t_k^*), t_k^*)\, \mathbf{w}(t_k^*)\, (t_{k+1} - t_k),$$

  where $t_0 < t_1 < \ldots < t_n = t$ and $t_k^* \in [t_k, t_{k+1}]$.
- Upper and lower sums are defined as the selections of t_k^* such that the integrand $\mathbf{L}(\mathbf{x}(t_k^*), t_k^*)\, \mathbf{w}(t_k^*)$ has its maximum and minimum values, respectively.
- The Riemann integral exists if the upper and lower sums converge to the same value.
- Because white noise is discontinuous everywhere, the Riemann integral does not exist.

Numerical approaches

Attempt 2: Stieltjes integral

- Stieltjes integral is more general than the Riemann integral.
- In particular, it allows for discontinuous integrands.
- We can interpret the increment $\mathbf{w}(t)\ \mathrm{d}t$ as increment of another process $\beta(t)$ such that

$$\int_{t_0}^{t} \mathbf{L}(\mathbf{x}(t), t)\, \mathbf{w}(t)\ \mathrm{d}t = \int_{t_0}^{t} \mathbf{L}(\mathbf{x}(t), t)\ \mathrm{d}\beta(t).$$

- It turns out that a suitable process for this purpose is the Brownian motion —

Numerical approaches

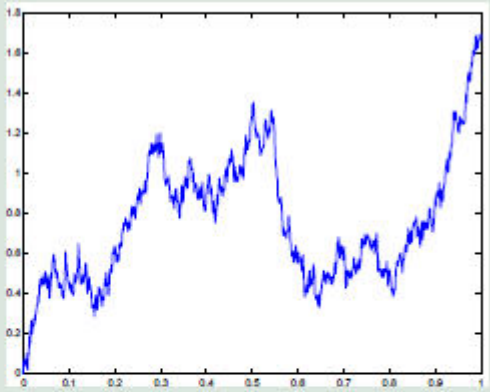
Brownian motion

1. Gaussian increments:

$$\Delta\beta_k \sim \mathrm{N}(0, \mathbf{Q}\,\Delta t_k),$$

where $\Delta\beta_k = \beta(t_{k+1}) - \beta(t_k)$ and $\Delta t_k = t_{k+1} - t_k$.

2. Non-overlapping increments are independent.



- **Q** is the diffusion matrix of the Brownian motion.
- Brownian motion $t \mapsto \beta(t)$ has discontinuous derivative everywhere.
- White noise can be considered as the formal derivative of Brownian motion $\mathbf{w}(t) = \mathrm{d}\beta(t)/\mathrm{d}t$.

Numerical approaches

Attempt 2: Stieltjes integral (cont.)

- Stieltjes integral is defined as a limit of the form

$$\int_{t_0}^{t} \mathbf{L}(\mathbf{x}(t), t)\, \mathrm{d}\beta = \lim_{n\to\infty} \sum_{k} \mathbf{L}(\mathbf{x}(t_k^*), t_k^*)\, [\beta(t_{k+1}) - \beta(t_k)],$$

  where $t_0 < t_1 < \ldots < t_n$ and $t_k^* \in [t_k, t_{k+1}]$.
- The limit t_k^* should be independent of the position on the interval $t_k^* \in [t_k, t_{k+1}]$.
- But for integration with respect to Brownian motion this is not the case.
- Thus, Stieltjes integral definition does not work either.

Numerical approaches

Attempt 3: Lebesgue integral

- In Lebesgue integral we could interpret $\beta(t)$ to define a "stochastic measure" via $\beta((u,v)) = \beta(u) - \beta(v)$.
- Essentially, this will also lead to the definition

$$\int_{t_0}^{t} \mathbf{L}(\mathbf{x}(t), t)\, \mathrm{d}\beta = \lim_{n\to\infty} \sum_{k} \mathbf{L}(\mathbf{x}(t_k^*), t_k^*)\, [\beta(t_{k+1}) - \beta(t_k)],$$

  where $t_0 < t_1 < \ldots < t_n$ and $t_k^* \in [t_k, t_{k+1}]$.
- Again, the limit should be independent of the choice $t_k^* \in [t_k, t_{k+1}]$.
- Also our "measure" is not really a sensible measure at all.
- $\Rightarrow$ Lebesgue integral does not work either.

Numerical approaches

Attempt 4: Itô integral

- The solution to the problem is the Itô stochastic integral.
- The idea is to fix the choice to $t_k^* = t_k$, and define the integral as

$$\int_{t_0}^{t} \mathbf{L}(\mathbf{x}(t), t)\, \mathrm{d}\beta(t) = \lim_{n\to\infty} \sum_k \mathbf{L}(\mathbf{x}(t_k), t_k)\, [\beta(t_{k+1}) - \beta(t_k)].$$

- This Itô stochastic integral turns out to be a sensible definition of the integral.
- However, the resulting integral does not obey the computational rules of ordinary calculus.
- Instead of ordinary calculus we have Itô calculus.

Numerical approaches

Itô stochastic differential equations

- Consider the white noise driven ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t)\,\mathbf{w}(t).$$

- This is actually defined as the Itô integral equation

$$\mathbf{x}(t) - \mathbf{x}(t_0) = \int_{t_0}^{t} \mathbf{f}(\mathbf{x}(t), t)\ \mathrm{d}t + \int_{t_0}^{t} \mathbf{L}(\mathbf{x}(t), t)\ \mathrm{d}\boldsymbol{\beta}(t),$$

  which should be true for arbitrary t_0 and t.

- Settings the limits to t and $t + \mathrm{d}t$, where $\mathrm{d}t$ is "small", we get

$$\mathrm{d}\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\ \mathrm{d}t + \mathbf{L}(\mathbf{x}, t)\ \mathrm{d}\boldsymbol{\beta}.$$

- This is the canonical form of an Itô SDE.

Numerical approaches

Connection with white noise driven ODEs

- Let's formally divide by $\mathrm{d}t$, which gives

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t)\,\frac{\mathrm{d}\beta}{\mathrm{d}t}.$$

- Thus we can interpret $\mathrm{d}\beta/\mathrm{d}t$ as white noise $\mathbf{w}$.
- Note that we cannot define more general equations

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}(t), \mathbf{w}(t), t),$$

because we cannot re-interpret this as an Itô integral equation.

- White noise should not be thought as an entity as such, but it only exists as the formal derivative of Brownian motion.

Numerical approaches

Taylor series of ODEs vs. Itô-Taylor series of SDEs

- Taylor series expansions (in time direction) are classical methods for approximating solutions of deterministic ordinary differential equations (ODEs).
- Largely superseded by Runge–Kutta type of derivative free methods (whose theory is based on Taylor series).
- Itô-Taylor series can be used for approximating solutions of SDEs—direct generalization of Taylor series for ODEs.
- Stochastic Runge–Kutta methods are not as easy to use as their deterministic counterparts
- It is easier to understand Itô-Taylor series by understanding Taylor series (for ODEs) first.

STOCHASTIC DEQ: FORMULATION

First, let us recall how we can obtain a Taylor expansion from an integral representation for the deterministic case. Consider the ODE,

$$\frac{d}{dt}X(t) = a[X(t)].$$

Let f be a function of $X(t)$, then the evolution of the function f is governed by

$$\frac{d}{dt}f[X(t)] = a[X(t)]\frac{\partial}{\partial X}f[X(t)] \tag{1}$$

via the chain rule.

STOCHASTIC DEQ: FORMULATION

Integral Equation: By defining a linear operator,

$$\mathcal{L} \equiv a(X) \frac{\partial}{\partial X},$$

we can rewrite Eq. (1) in terms of the integral equation:

$$f[X(t)] = f[X(t_0)] + \int_{t_0}^{t} \mathcal{L} f[X(\tau_1)] \, d\tau_1 \qquad (2)$$

where t_0 is the initial time.

STOCHASTIC DEQ: FORMULATION

Iterations: Iterating Eq. (2) leads to

$$
\begin{aligned}
f\left[X\left(t\right)\right] &= f\left[X\left(t_0\right)\right] + \int_{t_0}^{t} \mathcal{L}\left[f\left[X\left(t_0\right)\right] + \int_{t_0}^{\tau_1} \mathcal{L}f\left[X\left(\tau_2\right)\right] d\tau_2\right] d\tau_1 \\
&= f\left[X\left(t_0\right)\right] + \mathcal{L}f\left[X\left(t_0\right)\right] \int_{t_0}^{t} d\tau_1 + \int_{t_0}^{t}\int_{t_0}^{\tau_1} \mathcal{L}^2 f\left[X\left(\tau_2\right)\right] d\tau_2 d\tau_1 \\
&= f\left[X\left(t_0\right)\right] + \mathcal{L}f\left[X\left(t_0\right)\right]\left(t-t_0\right) + \int_{t_0}^{t}\int_{t_0}^{\tau_1} \mathcal{L}^2 f\left[X\left(\tau_2\right)\right] d\tau_2 d\tau_1
\end{aligned}
$$

If we iterate once more using Eq. (2) for $f\left(X\left(\tau_2\right)\right)$ in the integrand of the double integral,

$$
\begin{aligned}
\mathcal{L}^2 f\left[X\left(\tau_2\right)\right] &= \mathcal{L}^2\left[f\left[X\left(t_0\right)\right] + \int_{t_0}^{\tau_2} \mathcal{L}f\left[X\left(\tau_3\right)\right] d\tau_3\right] \\
&= \mathcal{L}^2 f\left[X\left(t_0\right)\right] + \underbrace{\mathcal{L}^2 \int_{t_0}^{\tau_2} \mathcal{L}f\left[X\left(\tau_3\right)\right] d\tau_3}_{O(\mathcal{L}^3)}
\end{aligned}
$$

STOCHASTIC DEQ: FORMULATION

If we iterate once more using Eq. (2) for $f\left(X\left(\tau_2\right)\right)$ in the integrand of the double integral,

$$\begin{aligned}\mathcal{L}^2 f\left[X\left(\tau_2\right)\right] &= \mathcal{L}^2\left[f\left[X\left(t_0\right)\right]+\int_{t_0}^{\tau_2}\mathcal{L}f\left[X\left(\tau_3\right)\right]d\tau_3\right] \\ &= \mathcal{L}^2 f\left[X\left(t_0\right)\right]+\underbrace{\mathcal{L}^2\int_{t_0}^{\tau_2}\mathcal{L}f\left[X\left(\tau_3\right)\right]d\tau_3}_{O(\mathcal{L}^3)}\end{aligned}$$

then the contribution of the first term above, which is a constant, to the double integral is

$$\begin{aligned}\int_{t_0}^{t}\int_{t_0}^{\tau_1}\mathcal{L}^2 f\left[X\left(t_0\right)\right]d\tau_2 d\tau_1 &= \mathcal{L}^2 f\left[X\left(t_0\right)\right]\int_{t_0}^{t}\int_{t_0}^{\tau_1}d\tau_2 d\tau_1 \\ &= \frac{1}{2}\mathcal{L}^2 f\left[X\left(t_0\right)\right]\left(t-t_0\right)^2\end{aligned}$$

Putting all these together, we have

$$f\left[X\left(t\right)\right]=f\left[X\left(t_0\right)\right]+\mathcal{L}f\left[X\left(t_0\right)\right]\left(t-t_0\right)+\frac{1}{2}\mathcal{L}^2 f\left[X\left(t_0\right)\right]\left(t-t_0\right)^2+O\left(\mathcal{L}^3\right)$$

which is precisely the Taylor expansion we are familiar with.

STOCHASTIC DEQ: FORMULATION

Ito-Taylor Expansion

Now we discuss a similar expansion for the stochastic differential equation:

$$dX(t) = a[X(t)]\,dt + b[X(t)]\,dW(t). \tag{3}$$

Again, for simplicity, we consider the autonomous case, i.e., $a = a[X(t)]$, $b = b[X(t)]$ and they do not depend on time explicitly.

The Ito lemma leads to

$$df[X(t)] = \left\{ a\frac{\partial}{\partial X} f[X(t)] + \frac{1}{2} b^2[X(t)] \frac{\partial^2}{\partial X^2} f[X(t)] \right\} dt + b[X(t)] \frac{\partial}{\partial X} f[X(t)]\, dW(t) \tag{4}$$

Defining

$$\begin{aligned} \mathcal{L}^0 &\equiv a\frac{\partial}{\partial X} + \frac{1}{2} b^2[X] \frac{\partial^2}{\partial X^2} \\ \mathcal{L}^1 &\equiv b[X] \frac{\partial}{\partial X} \end{aligned}$$

STOCHASTIC DEQ: FORMULATION

then Eq. (4) becomes

$$df\left[X\left(t\right)\right]=\mathcal{L}^{0}f\left[X\left(t\right)\right]dt+\mathcal{L}^{1}f\left[X\left(t\right)\right]dW\left(t\right)$$

i.e.,

$$f\left[X\left(t\right)\right]=f\left[X\left(t_{0}\right)\right]+\int_{t_{0}}^{t}\mathcal{L}^{0}f\left[X\left(s\right)\right]ds+\int_{t_{0}}^{t}\mathcal{L}^{1}f\left[X\left(s\right)\right]dW\left(s\right) \tag{5}$$

In equation (5), if we choose the following $f\left(x\right)$:

1. Choose $f\left(x\right)=x$, then Eq. (5) becomes

$$X\left(t\right)=X\left(t_{0}\right)+\int_{t_{0}}^{t}a\left[X\left(s\right)\right]ds+\int_{t_{0}}^{t}b\left[X\left(s\right)\right]dW\left(s\right) \tag{6}$$

2. Choose $f\left(x\right)=a\left(x\right)$, then Eq. (5) becomes

$$a\left[X\left(t\right)\right]=a\left[X\left(t_{0}\right)\right]+\int_{t_{0}}^{t}\mathcal{L}^{0}a\left[X\left(s\right)\right]ds+\int_{t_{0}}^{t}\mathcal{L}^{1}a\left[X\left(s\right)\right]dW\left(s\right) \tag{7}$$

STOCHASTIC DEQ: FORMULATION

2. Choose $f(x) = a(x)$, then Eq. (5) becomes

$$a[X(t)] = a[X(t_0)] + \int_{t_0}^{t} \mathcal{L}^0 a[X(s)]\, ds + \int_{t_0}^{t} \mathcal{L}^1 a[X(s)]\, dW(s) \tag{7}$$

3. Similarly, choose $f(x) = b(x)$, then Eq. (5) becomes

$$b[X(t)] = b[X(t_0)] + \int_{t_0}^{t} \mathcal{L}^0 b[X(s)]\, ds + \int_{t_0}^{t} \mathcal{L}^1 b[X(s)]\, dW(s) \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6) leads to

$$\begin{aligned} X(t) &= X(t_0) + \int_{t_0}^{t} \left\{ a[X(t_0)] + \int_{t_0}^{s_1} \mathcal{L}^0 a[X(s_2)]\, ds_2 + \int_{t_0}^{s_1} \mathcal{L}^1 a[X(s_2)]\, dW(s_2) \right\} ds_1 \\ &\quad + \int_{t_0}^{t} \left\{ b[X(t_0)] + \int_{t_0}^{s_1} \mathcal{L}^0 b[X(s_2)]\, ds_2 + \int_{t_0}^{s_1} \mathcal{L}^1 b[X(s_2)]\, dW(s_2) \right\} dW(s_1) \end{aligned} \tag{9}$$

STOCHASTIC DEQ: FORMULATION

Note that

$$\begin{aligned}
\mathcal{L}^0 a &= a\frac{\partial}{\partial X}a + \frac{1}{2}b^2\left[X\right]\frac{\partial^2}{\partial X^2}a \equiv aa' + \frac{1}{2}b^2 a'' \\
\mathcal{L}^0 b &= ab' + \frac{1}{2}b^2 b'' \\
\mathcal{L}^1 a &= b\frac{\partial}{\partial X}a = ba' \\
\mathcal{L}^1 b &= bb'
\end{aligned}$$

Separating the constant terms out in the integrand in Eq. (9) from the remaining terms, which are the double integral terms:

$$\begin{aligned}
R \equiv & \int_{t_0}^{t}\int_{t_0}^{s_1} \mathcal{L}^0 a\left[X\left(s_2\right)\right] ds_2 ds_1 + \int_{t_0}^{t}\int_{t_0}^{s_1} \mathcal{L}^1 a\left[X\left(s_2\right)\right] dW\left(s_2\right) ds_1 \\
& + \int_{t_0}^{t}\int_{t_0}^{s_1} \mathcal{L}^0 b\left[X\left(s_2\right)\right] ds_2 dW\left(s_1\right) + \int_{t_0}^{t}\int_{t_0}^{s_1} \mathcal{L}^1 b\left[X\left(s_2\right)\right] dW\left(s_2\right) dW\left(s_1\right),
\end{aligned}$$

leads to

$$X\left(t\right) = X\left(t_0\right) + a\left[X\left(t_0\right)\right]\int_{t_0}^{t} ds_1 + b\left[X\left(t_0\right)\right]\int_{t_0}^{t} dW\left(s_1\right) + R \qquad (10)$$

STOCHASTIC DEQ: FORMULATION

Note that the essence of the method is to use the substitution repeatedly to obtain constant integrands in higher and higher order terms. For example, the last term in the remainder, R, (which is of the lowest order in Δt in R if $\Delta t \equiv t - t_0$ is small) is

$$\int_{t_0}^{t}\int_{t_0}^{s_1} \mathcal{L}^1 b\left[X\left(s_2\right)\right] dW\left(s_2\right) dW\left(s_1\right)$$

$$= \int_{t_0}^{t}\int_{t_0}^{s_1} \left\{ \mathcal{L}^1 b\left[X\left(t_0\right)\right] + \int_{t_0}^{s_2} \mathcal{L}^0\mathcal{L}^1 b\left[X\left(s_3\right)\right] ds_3 + \int_{t_0}^{s_2} \mathcal{L}^1\mathcal{L}^1 b\left[X\left(s_3\right)\right] dW\left(s_3\right) \right\} dW\left(s_2\right) dW\left(s_1\right)$$

by selecting $f = \mathcal{L}^1 b$ in Eq. (5). The first term in the last line of the above equation is

$$b\left[X\left(t_0\right)\right] b'\left[X\left(t_0\right)\right] \int_{t_0}^{t}\int_{t_0}^{s_1} dW\left(s_2\right) dW\left(s_1\right)$$

by noting $\mathcal{L}^1 b = bb'$. Therefore, Eq. (10) becomes

$$\begin{aligned} X\left(t\right) &= X\left(t_0\right) + a\left[X\left(t_0\right)\right] \int_{t_0}^{t} ds_1 + b\left[X\left(t_0\right)\right] \int_{t_0}^{t} dW\left(s_1\right) \\ &\quad + b\left[X\left(t_0\right)\right] b'\left[X\left(t_0\right)\right] \int_{t_0}^{t}\int_{t_0}^{s_1} dW\left(s_2\right) dW\left(s_1\right) + \tilde{R} \end{aligned} \qquad (11)$$

where $\tilde{R}$ is a new remainder. Eq. (11) is the Ito-Taylor expansion for the process (3).

STOCHASTIC DEQ: FORMULATION

Note that the double integral in Eq. (11) is evaluated to be

$$\int_{t_0}^{t}\int_{t_0}^{s_1} dW(s_2)\,dW(s_1) = \frac{1}{2}\left[W(t) - W(t_0)\right]^2 - \frac{1}{2}(t - t_0),$$

which can be shown as follows

$$\begin{aligned}
\int_{t_0}^{t}\int_{t_0}^{s_1} dW(s_2)\,dW(s_1) &= \int_{t_0}^{t}\left[W(s_1) - W(t_0)\right]dW(s_1) \\
&= \int_{t_0}^{t} W(s_1)\,dW(s_1) - \int_{t_0}^{t} W(t_0)\,dW(s_1) \\
&= \frac{1}{2}\int_{t_0}^{t}\left[dW^2(s_1) - dt\right] - \int_{t_0}^{t} W(t_0)\,dW(s_1) \\
&\qquad\qquad \left(\because dW^2 = 2W dW + dt \quad \text{(Ito Lemma)}\right) \\
&= \frac{1}{2}\left[W^2(t) - W^2(t_0) - (t - t_0)\right] - W(t_0)\left[W(t) - W(t_0)\right] \\
&= \frac{1}{2}\left[W(t) - W(t_0)\right]^2 - \frac{1}{2}(t - t_0) \qquad (12)
\end{aligned}$$

SDEQ: Numerical Integration Schemes

Once we have the Ito-Taylor expansion, we can construct numerical integration schemes for the process (3). For the interval $[t_i, t_{i+1}]$, by choosing

$$\begin{aligned} t_0 &= t_i, \quad t = t_{i+1}, \\ \Delta t &= t_{i+1} - t_i \\ \Delta W_i &= W(t_{i+1}) - W(t_i), \end{aligned}$$

combining Eq. (11) and Eq. (12) yields

$$X(t_{i+1}) = X(t_i) + a[X(t_i)]\Delta t + b[X(t_i)]\Delta W_i + \frac{1}{2} b[X(t_i)] b'[X(t_i)] \left[(\Delta W_i)^2 - \Delta t\right] + \tilde{R}. \tag{13}$$

1. Keeping the first three terms in Eq. (13) gives us the explicit Euler method:

$$\hat{X}_{i+1} = \hat{X}_i + a\left[\hat{X}_i\right]\Delta t + b\left[\hat{X}_i\right]\Delta W_i$$

2. Keeping all terms of $O(\Delta t)$ gives us the Milstein method:

$$\hat{X}_{i+1} = \hat{X}_i + a\left[\hat{X}_i\right]\Delta t + b\left[\hat{X}_i\right]\Delta W_i + \frac{1}{2} b\left[\hat{X}_i\right] b'\left[\hat{X}_i\right]\left[(\Delta W_i)^2 - \Delta t\right] \tag{14}$$

INTRODUCTION TO STOCHASTIC SIMULATION

Simulation of Brownian motion

- W(0)=0 with Probability 1
- $$W_j = W_{j-1} + dW_j, j = 1, 2, \ldots, N$$
- Each dWj is an independent random variable of the form $\sqrt{\delta t}N(0,1)]$
- random number generator randn – each call to randn produces an "pseudorandom" number
- Notice: Matlab starts arrays from index 1 and not index 0.

BROWNIAN MOTION SIMULATION

```
randn('state',100)              % set the state of randn

T=1; N=500; dt=T/N;

dW=zeros(1,N);                  % preallocate arrays
W=zeros(1,N);                   % for efficiency
dW(1)=sqrt(dt)*randn;           % first approximation outside the loop

W(1)=dW(1);                     % since W(0)=0 is not allowed

for   j = 2:N
      dW(j)= sqrt(dt)*randn;          % general increment
      W(j)=W(j-1) + dW(j);
end

plot([0:dt:T],[0,W],'r-');          %   plot W against t

xlabel('t','FomtSize',16)
ylabel('W(t),'FontSize',16,'Rotation',0)
```

BROWNIAN MOTION SIMULATION

```
randn('state',100)     % set the state of randn

T = 1; N = 500; dt = T/N;

dW = sqrt(dt)*randn(1,N); % increments

W = cumsum(dW);          % cumulative sum

plot([0:dt:T],[0,W],'r-')      % plot W against t

xlabel('t','FontSize',16)

ylabel('W(t)','FontSize',16,'Rotation',0)
```

APPROXIMATION OF STOCHASTIC INTEGRALS

Given a suitable function h, consider two sums

- Ito integral

$$\sum_{i=0}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j))$$

- Stratonovich integral

$$\sum_{i=0}^{N-1} h(\frac{t_j + t_{j+1}}{2})(W(t_{j+1}) - W(t_j))$$

INTRODUCTION TO STOCHASTIC SIMULATION

```
%Approximate stochastic integrals
%% Ito and Stratonovich integrals of WdW
randn(state,100)        % set the state of randn
T = 1; N = 5 0; dt = T/N;
dW = sqrt(dt)*randn(1,N); % increments
W = cumsum(dW);         % cumulative sum
ito = sum([0,W(1 : end – 1)]. * dW)
strat = sum((0.5 * ([0,W(1 :end – 1)] +W) + 0.5* sqrt(dt)* randn(1,N)). * dW)
itoerr = abs(ito – 0.5 * (W(end)^2 – T))
straterr = abs(strat – 0.5*W(end)^2)
```

INTRODUCTION TO STOCHASTIC SIMULATION

A SDE is given by

- Integral form

$$X(t) = X_0 + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW(s), \quad 0 \leq t \leq T.$$

- Or differential equation form

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(0) = X_0, \quad 0 \leq t \leq T.$$

- The Euler-Maruyama(EM) method takes the form

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})(W(\tau_j) - W(\tau_{j-1})), \quad j = 1, 2, \ldots, L.$$

INTRODUCTION TO STOCHASTIC SIMULATION

Example: $f(X) = \lambda X$ and $g = \mu X$

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t)$$

- The exact solution to this SDE is

$$X(t) = X(0)exp((\lambda - \frac{1}{2}\mu^2)t + \mu W(t))$$

- $\lambda = 2$, $\mu = 1$ and X(0)=1

APPROXIMATION OF STOCHASTIC INTEGRALS

```
%EM Euler-Maruyama method on linear SDE
%
% SDE is dX = lambda *Xdt + mu*XdW, X(0) = Xzero,
% where lambda = 2, mu = 1 and Xzero = 1.
%
% Discretized Brownian path over [0,1] has dt = 2^ – 8.
% Euler-Maruyama uses timestep R *dt.

randn(state,100)

lambda = 2; mu = 1; Xzero = 1; % problem parameters

T = 1;N = 2^8; dt = 1/N;
dW = sqrt(dt) * randn(1,N);     % Brownian increments
W = cumsum(dW);                 % discretized Brownian path
Xtrue = Xzero * exp((lambda – 0.5 *mu^2) * ([dt : dt : T]) + mu *W);
plot([0:dt:T],[Xzero,Xtrue],'m-'), hold on
```

APPROXIMATION OF STOCHASTIC INTEGRALS

```
R = 4; Dt = R *dt; L = N/R;      % L EM steps of size Dt = R*dt
Xem = zeros(1,L);                % preallocate for efficiency
Xtemp = Xzero;

for j = 1:L
      Winc = sum(dW(R * (j − 1) + 1 : R * j ));
      Xtemp = Xtemp + Dt *lambda *Xtemp + mu *Xtemp *Winc;
      Xem(j) = Xtemp;
end

plot([0:Dt:T],[Xzero,Xem],'r–*'), hold off

xlabel('t','FontSize',12)

ylabel('X','FontSize',16,'Rotation',0,'HorizontalAlignment','right')

emerr = abs(Xem(end) – Xtrue(end))
```

STOCHASTIC BOD PROBLEM

Consider the stochastic BOD model

$$\frac{dB_t}{dt} = -K_1 B_t + s_1 - B_t \sigma \xi_t, \quad B_{t_0} = B_0$$

or in terms of the Wiener process:

$$dB_t = (-K_1 B_t + s_1)\, dt - B_t \sigma dW_t, \quad B_{t_0} = B_0$$

Since the white noise process in this stochastic model is a mathematical approximation of a noise process with a relatively short correlation scale, this SDE has to be interpreted in the Stratonovich sense. Since the Euler scheme can only be used for Itô equations, the model above is rewritten as an Itô SDE:

$$dB_t = \left(-K_1 B_t + s_1 + \frac{1}{2} B_t \sigma^2\right) dt - B_t \sigma dW_t, \quad B_{t_0} = B_0$$

Refer to my website for the detailed m file

```
nsim=50;
N=100; % number of samples

mean=zeros(N,1); var=zeros(N,1);

for jj=1:nsim

dt=1; %time stepsize
%=======Initialization of the column vector=======
Rn=randn(N,1); t=zeros(N,1); w=zeros(N,1); dW=zeros(1,N);
%========Initial conditions=====================
 t(1)=1;

dW(1)=sqrt(dt)*randn;          % first approximation outside the loop

w(1)=dW(1);                    % since W(0)=0 is not allowed

 for i=2:100
    t(i)=i;
    w(i)=w(i-1)+sqrt(dt)*Rn(i);
    dW(i)= sqrt(dt)*Rn(i);
end

%declaraton of parameters
k1=0.1; s1=1; b=0.01;

%the numerical solution of the BOD model with the Euler scheme
B0=20.0;
B(1)=B0-k1*B0*dt+s1*dt+0.5*b*b*dt-B0*b*sqrt(dt)*dW(1);
   for i=2:100
            B(i)=B(i-1)-k1*B(i-1)*dt+s1*dt+0.5*B(i-1)*b*b*dt-b*sqrt(dt)*dW(i);
    end
Bplot(:,jj)=B;

%evaluation of the mean and variance of N samples of BOD
mean=mean+B'; var=var+B'.*B';
end
```