CE 609

LECTURE : Fast Fourier transform

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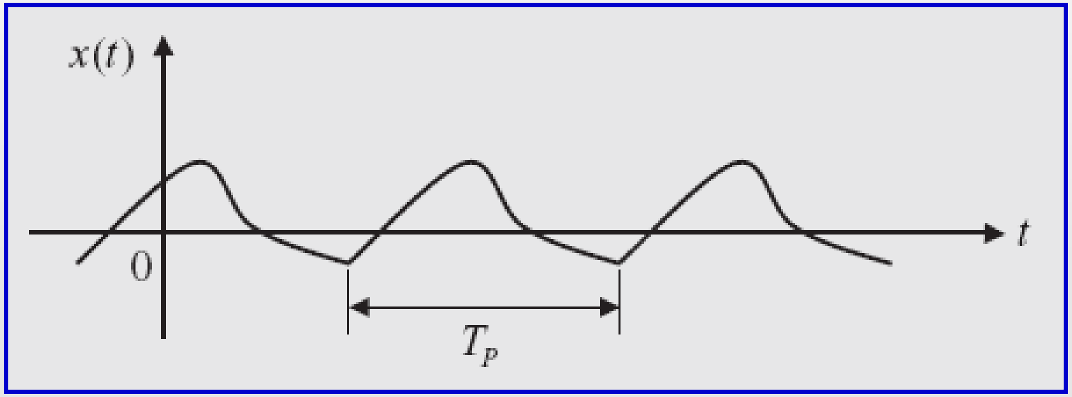
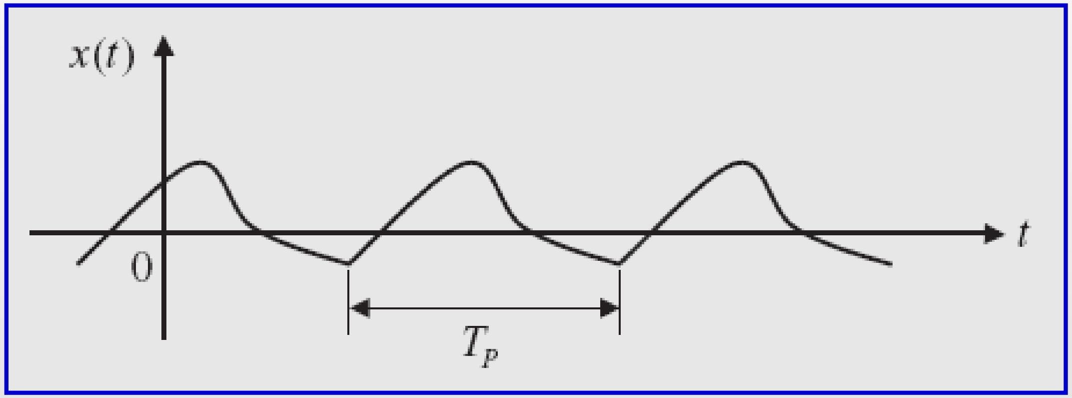
Fourier series



A period signal with a period T_P

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_P}\right) + b_n \sin\left(\frac{2\pi nt}{T_P}\right)\right]$$

Fourier series



$$\frac{a_0}{2} = \frac{1}{T_P}\int_0^{T_P} x(t)dt = \frac{1}{T_P}\int_{-T_P/2}^{T_P/2} x(t)dt \ : \quad \text{mean value}$$

$$a_n = \frac{2}{T_P}\int_0^{T_P} x(t)\cos\left(\frac{2\pi nt}{T_P}\right)dt = \frac{2}{T_P}\int_{-T_P/2}^{T_P/2} x(t)\cos\left(\frac{2\pi nt}{T_P}\right)dt \quad n = 1,\ 2,\ \ldots$$

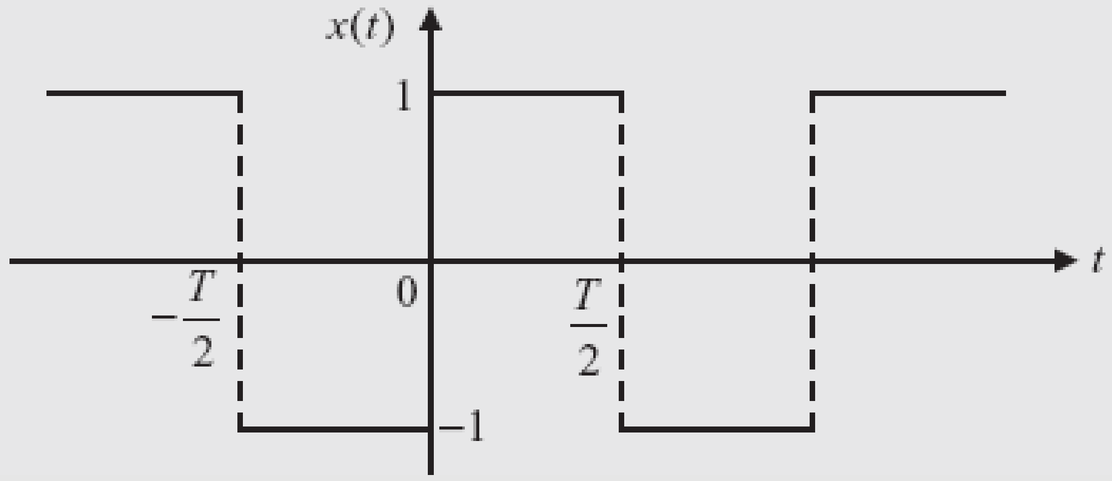
$$b_n = \frac{2}{T_P}\int_0^{T_P} x(t)\sin\left(\frac{2\pi nt}{T_P}\right)dt = \frac{2}{T_P}\int_{-T_P/2}^{T_P/2} x(t)\sin\left(\frac{2\pi nt}{T_P}\right)dt \quad n = 1,\ 2,\ \ldots$$

Fourier series: Square wave

$$x(t) = -1 \quad -\frac{T}{2} < t < 0$$

$$= 1 \quad 0 < t < \frac{T}{2}$$

and $\quad x(t + nT) = x(t) \quad n = \pm 1, \pm 2, \ldots$



Fourier series

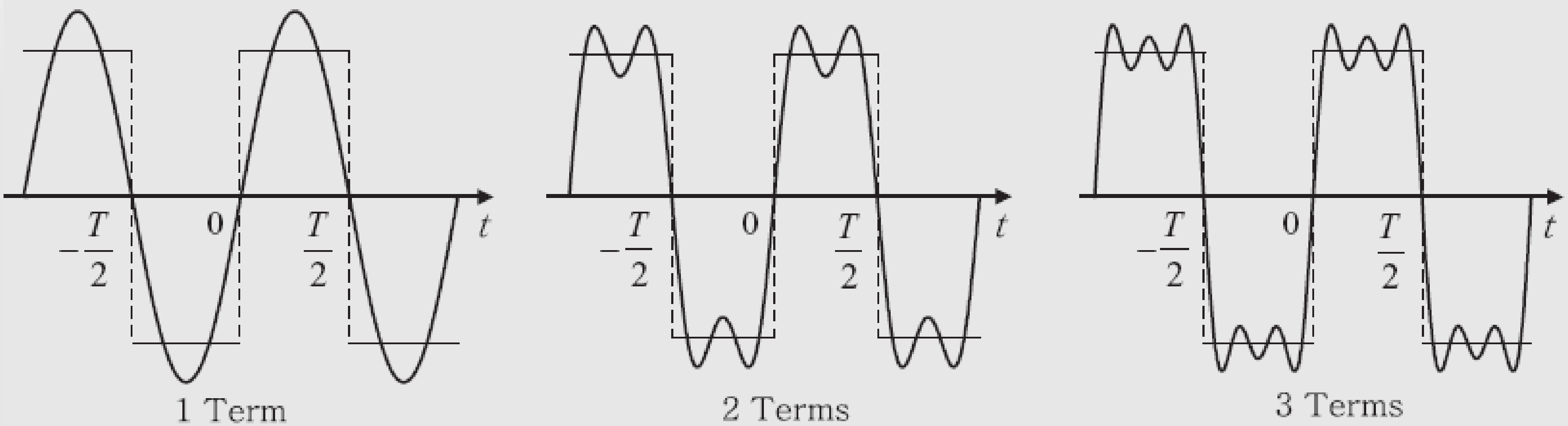
$$a_0 = 0; \quad a_n = 0$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin\left(\frac{2n\pi t}{T}\right) dt = \frac{2}{n\pi}\ (1 - n\pi)$$

$$x(t) = \frac{4}{\pi}\left[\sin\left(\frac{2\pi t}{T}\right) + \frac{1}{3}\sin\left(\frac{2\pi\, 3t}{T}\right) + \frac{1}{5}\sin\left(\frac{2\pi\, 5t}{T}\right) + \cdots\right]$$

$$x(t) = \frac{4}{\pi}\left[\sin \omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t + \cdots\right]$$

Fourier series



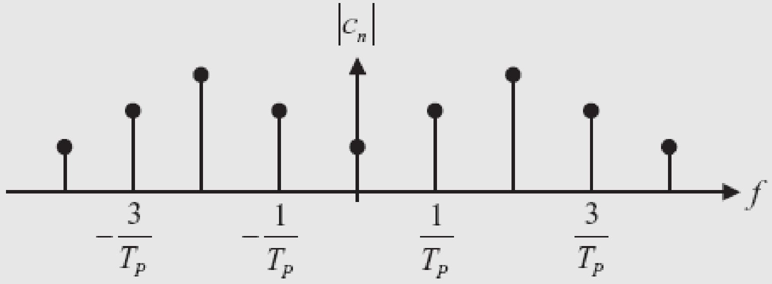
$$S_1(t) = \frac{4}{\pi}\sin\omega_1 t$$

$$S_2(t) = \frac{4}{\pi}\left[\sin\omega_1 t + \frac{1}{3}\sin 3\omega_1 t\right]$$

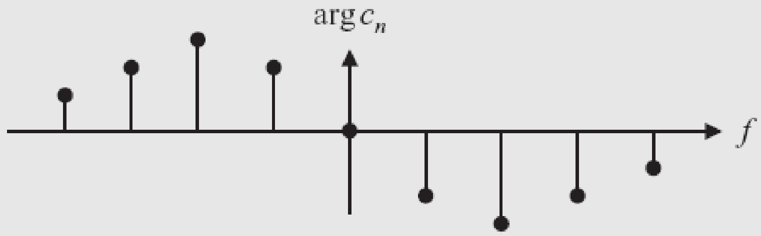
$$S_3(t) = \frac{4}{\pi}\left[\sin\omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t\right]$$

Fourier series-amplitude spectra

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{\frac{j2\pi nt}{T}} \qquad c_n = \frac{2}{T}\int_0^T x(t)\; e^{-\frac{j2\pi nt}{T}}\; dt$$



Amplitude spectrum of a Fourier series (a *line spectrum* and an *even function*)



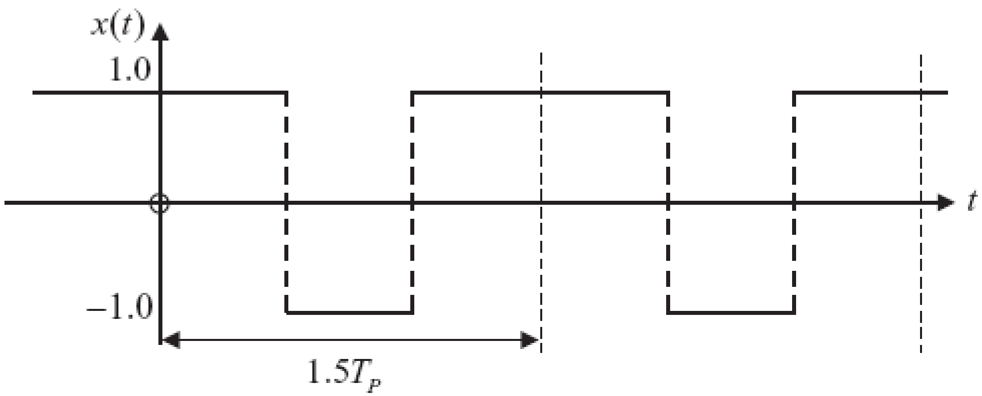
Phase spectrum of a Fourier series (a *line spectrum* and an *odd function*)

MATLAB example

```
clc; close all; clear all

 % Keep hitting enter button if you want to see the term by term
approximation
 t=[0:0.001:1];

x=[]; x_tmp=zeros(size(t));

for n=1:2:39
x_tmp=x_tmp+4/pi*(1/n*sin(2*pi*n*t));
x=[x; x_tmp];
end
 figure,
for i=1:20
    drawnow
        plot(t,x(i,:))   %plot(t,x(i,:),t,x(7,:),t,x(20,:));
        xlabel('\itt\rm (seconds)'); ylabel('\itx\rm(\itt\rm)')
        grid on
 pause
end
```

MATLAB exercise: Square wave

$$
\left.\begin{aligned}
|c_n| &= \frac{2}{n\pi} && \text{for } n = \text{odd} \\
&= 0 && \text{for } n = 0,\ \text{even}
\end{aligned}\right\}
$$



Examine the Fourier coefficients of the square wave for ($T_p = 1\ s$)

1) r an integer number
2) r : a fractional number

$$c_n = \frac{1}{rT_P} \int_0^{rT_P} x(t) e^{-j\frac{2\pi n}{rT_P}t}\, dt$$

Fourier series to Fourier transform

- Extension of Fourier analysis to non-periodic phenomena
- Discrete to continuous
- Skipping essential steps, in the limit $T_p \rightarrow \infty$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

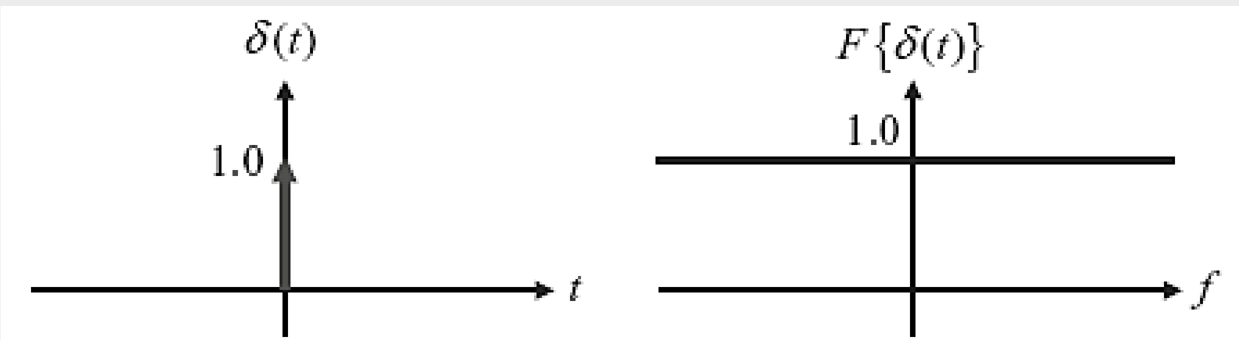
Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

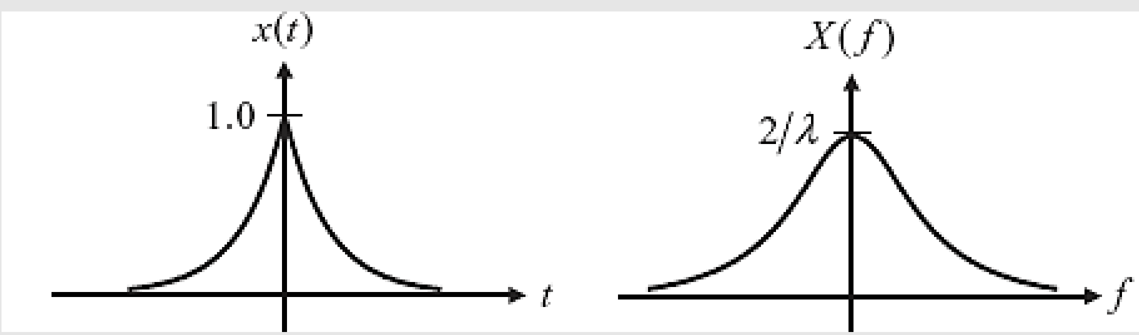
Inverse Fourier transform

Some examples: Try to commit some of them to memory, it helps !

1) Dirac delta



2) Symmetric exponential $x(t) = e^{-\lambda |t|}, \quad \lambda > 0$

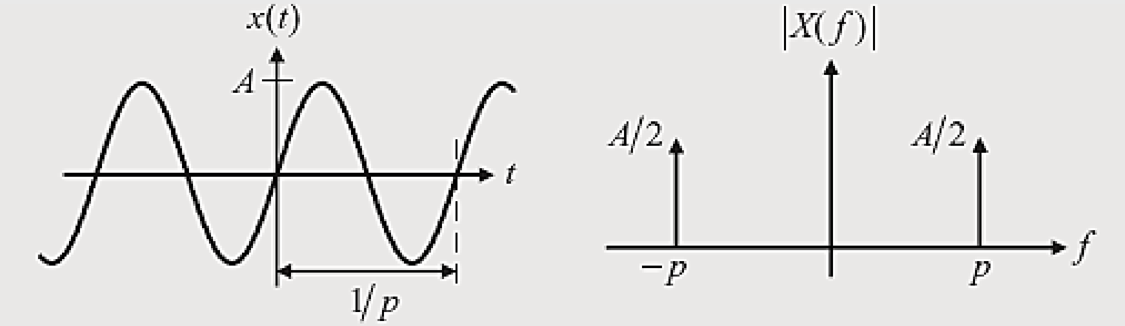


$$X(f) = \frac{2\lambda}{\lambda^2 + 4\pi^2 f^2}$$

Some examples

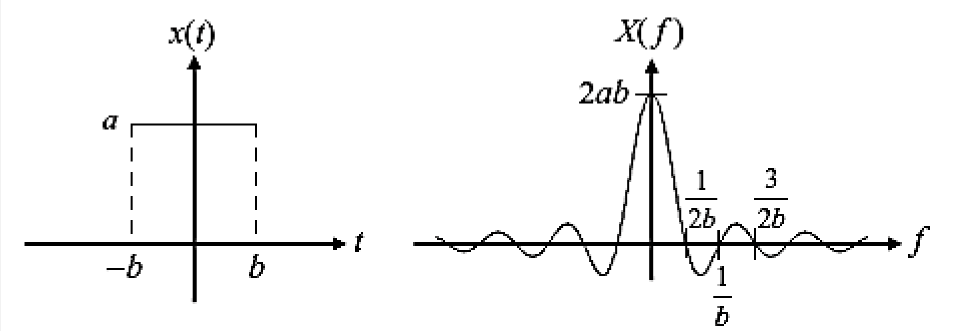
3) Sinusoid

$$x(t) = \sin(2\pi f_0 t) \text{ or } \sin(\omega_0 t) \quad X(f) = \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$



4) Window function

$$x(t) = a \quad |t| < b$$
$$= 0 \quad |t| > b$$



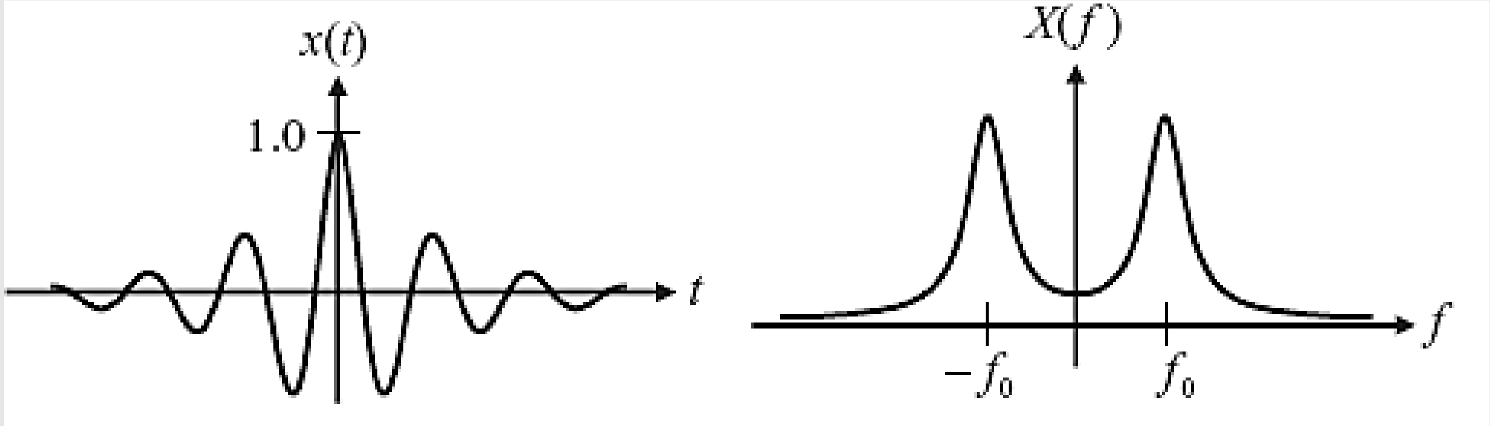
$$X(f) = \frac{2ab\sin(2\pi f b)}{2\pi f b}$$

Some examples

5) Damped symmetrically oscillating function

$$x(t) = e^{-a|t|} \cos 2\pi f_0 t, \quad a > 0$$

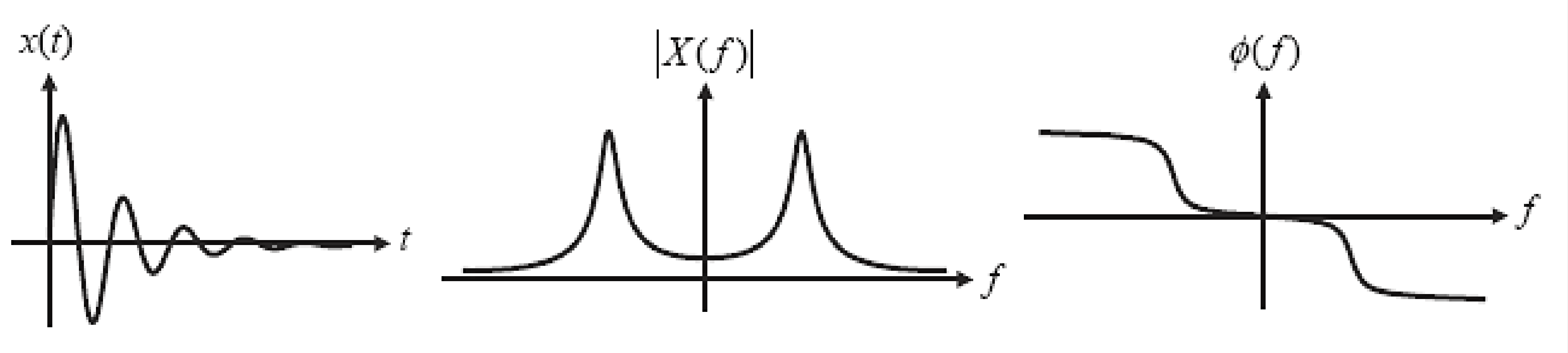
$$X(f) = \frac{a}{a^2 + [2\pi(f - f_0)]^2} + \frac{a}{a^2 + [2\pi(f + f_0)]^2}$$



6) Damped oscillating function

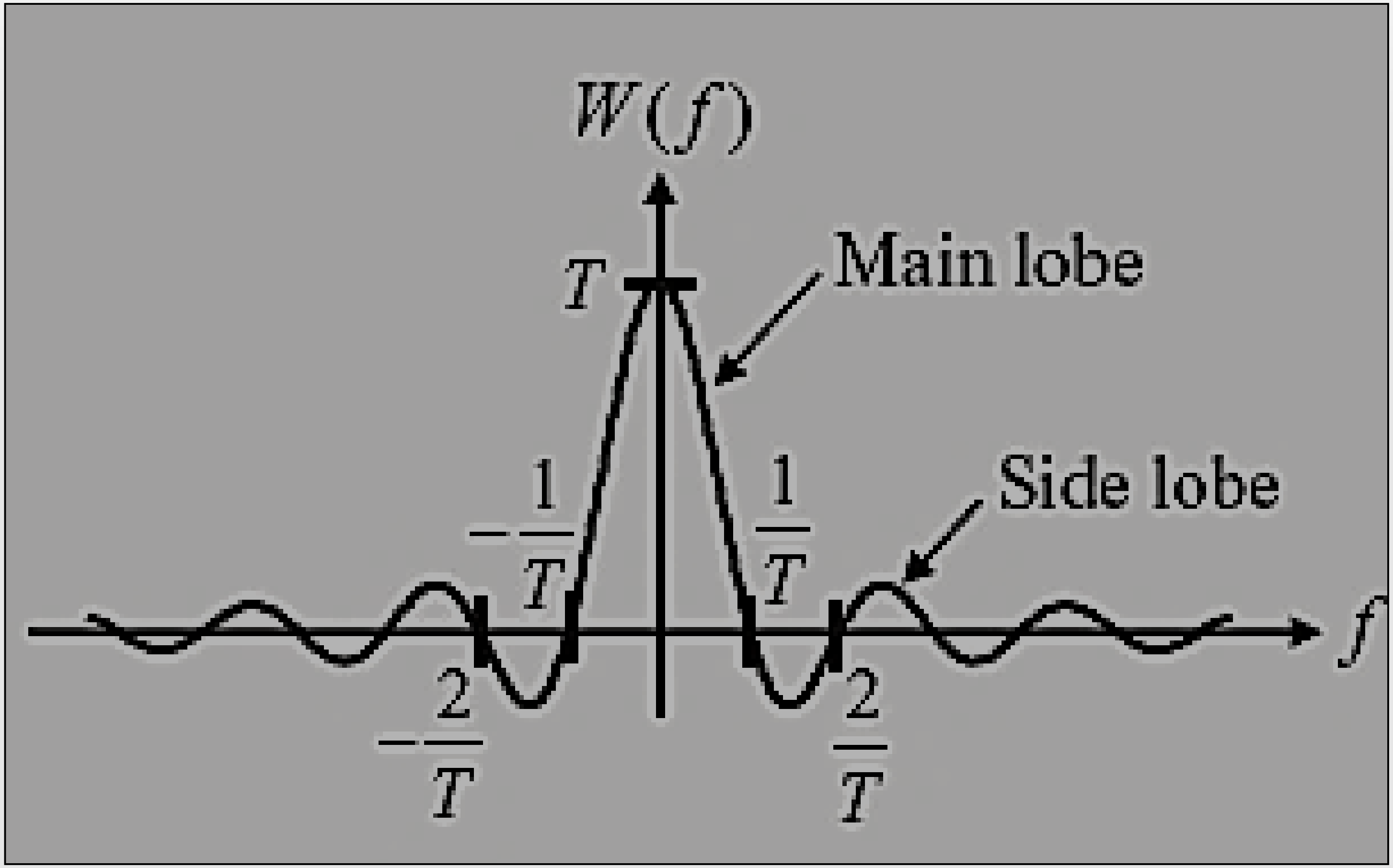
$$x(t) = e^{-at} \sin 2\pi f_0 t, \quad t \geq 0 \text{ and } a > 0$$

$$X(f) = \frac{2\pi f_0}{(2\pi f_0)^2 + (a + j2\pi f)^2}$$

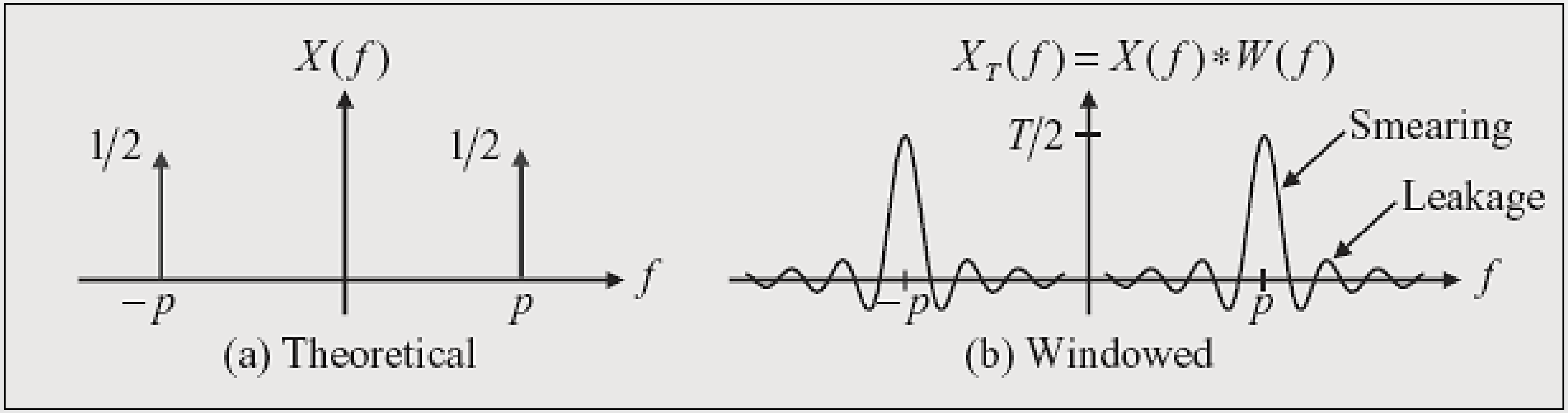


Windowing

Fourier transform of the rectangular window

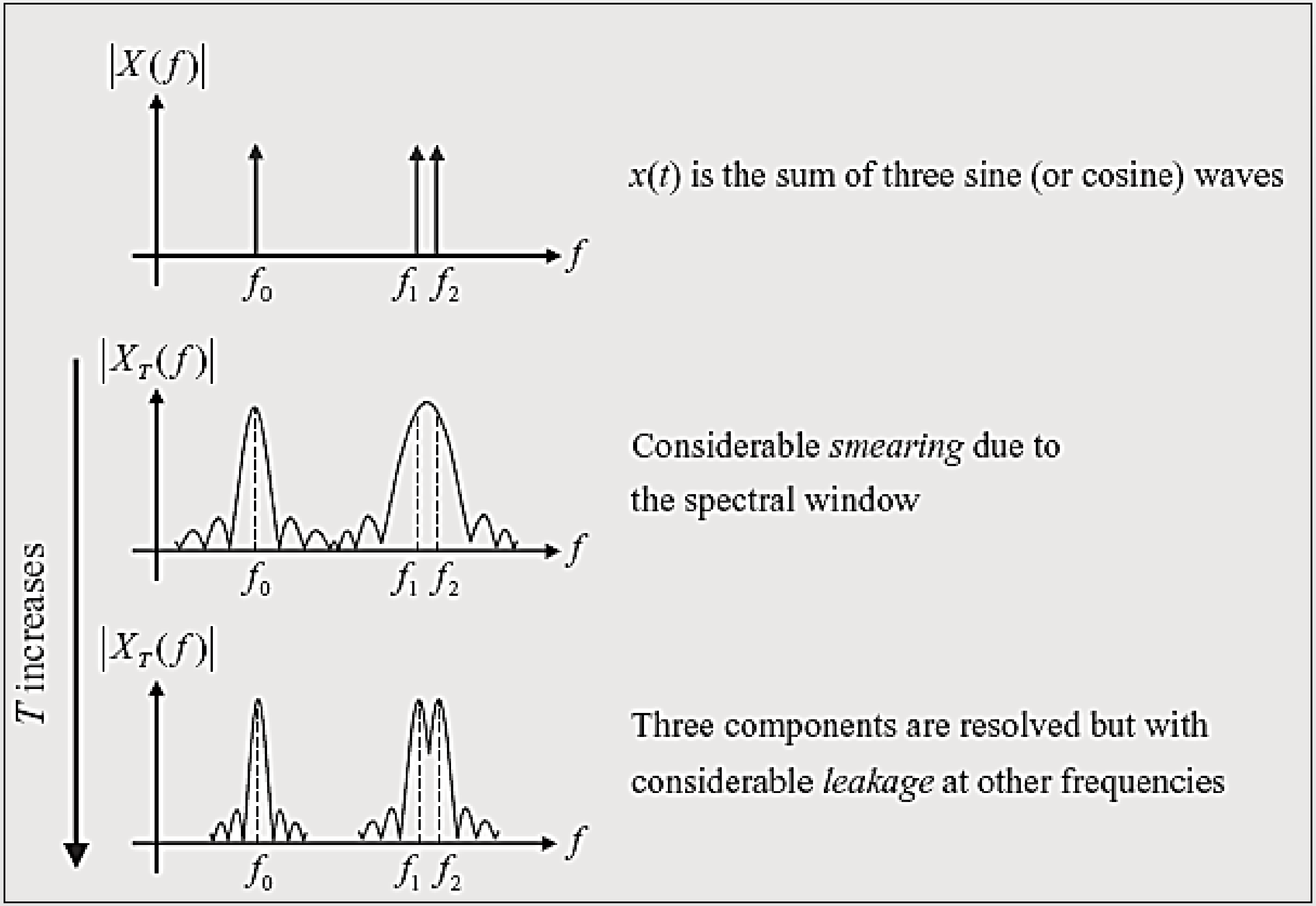


Windowing: a simple illustration



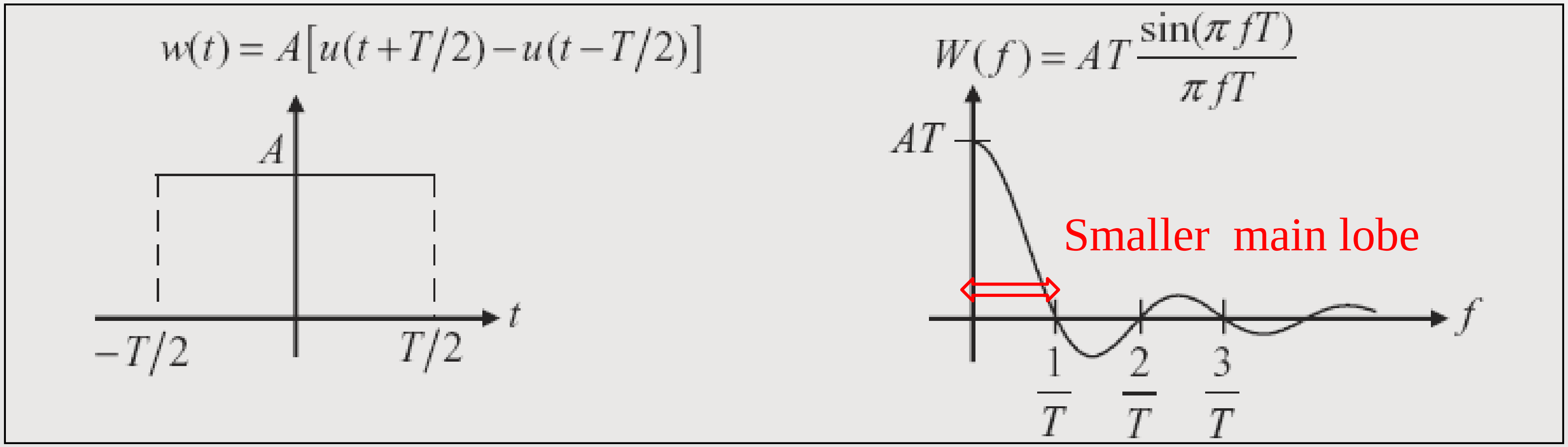
The distortion due to the main lobe is sometimes called *smearing*, and the distortion caused by the side lobes is called *leakage*.

Effects of windowing

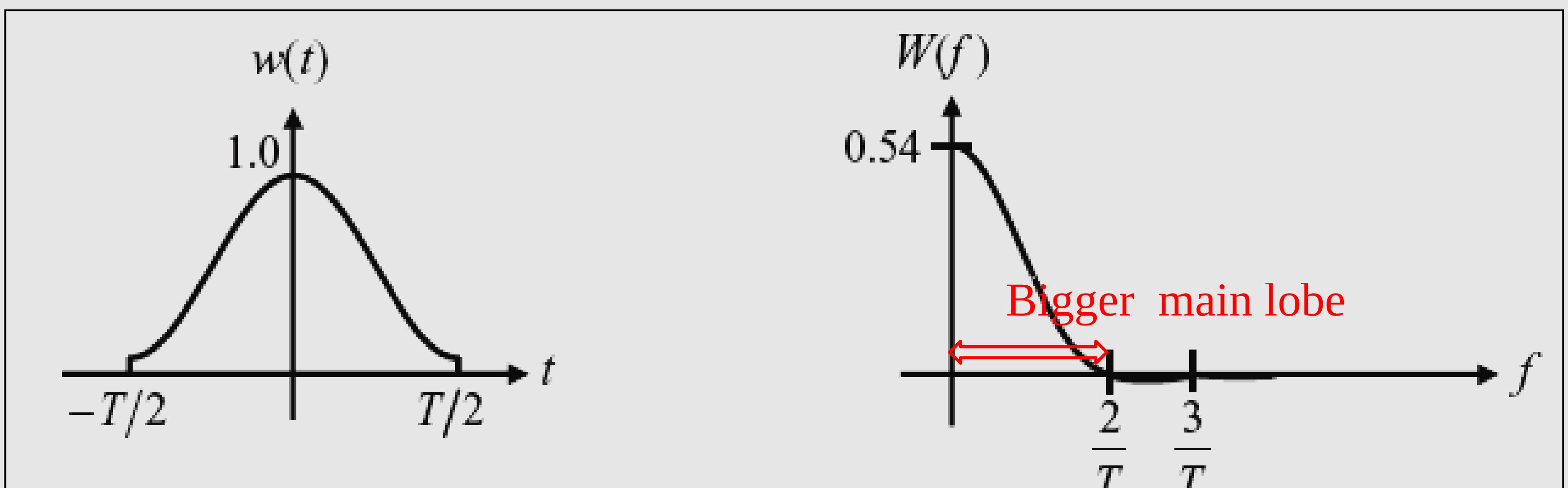


Common window functions

Rectangular Window

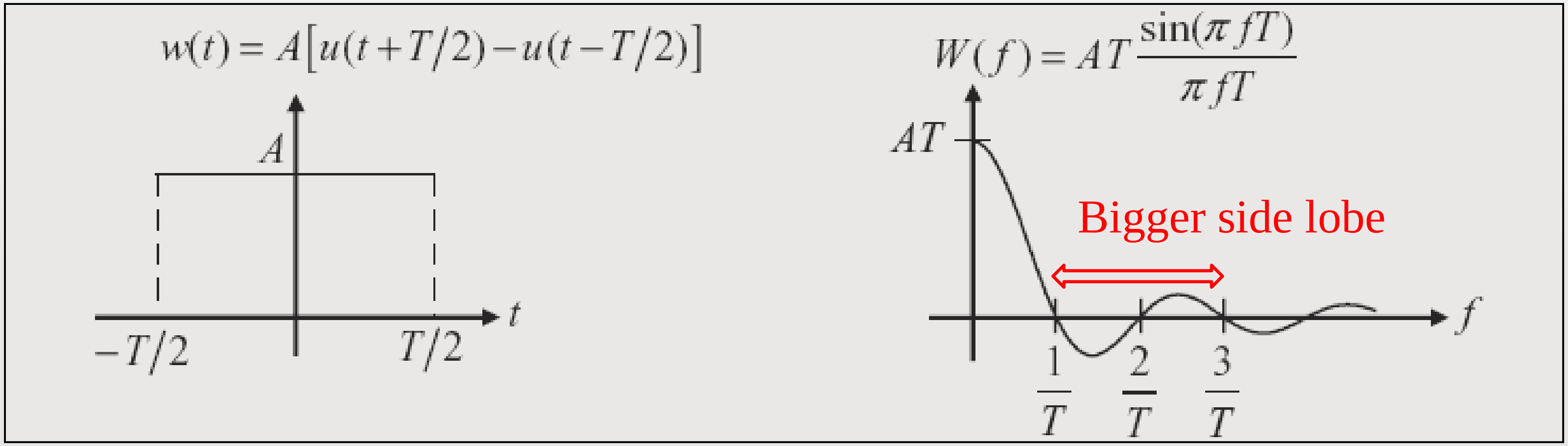


Hann Window

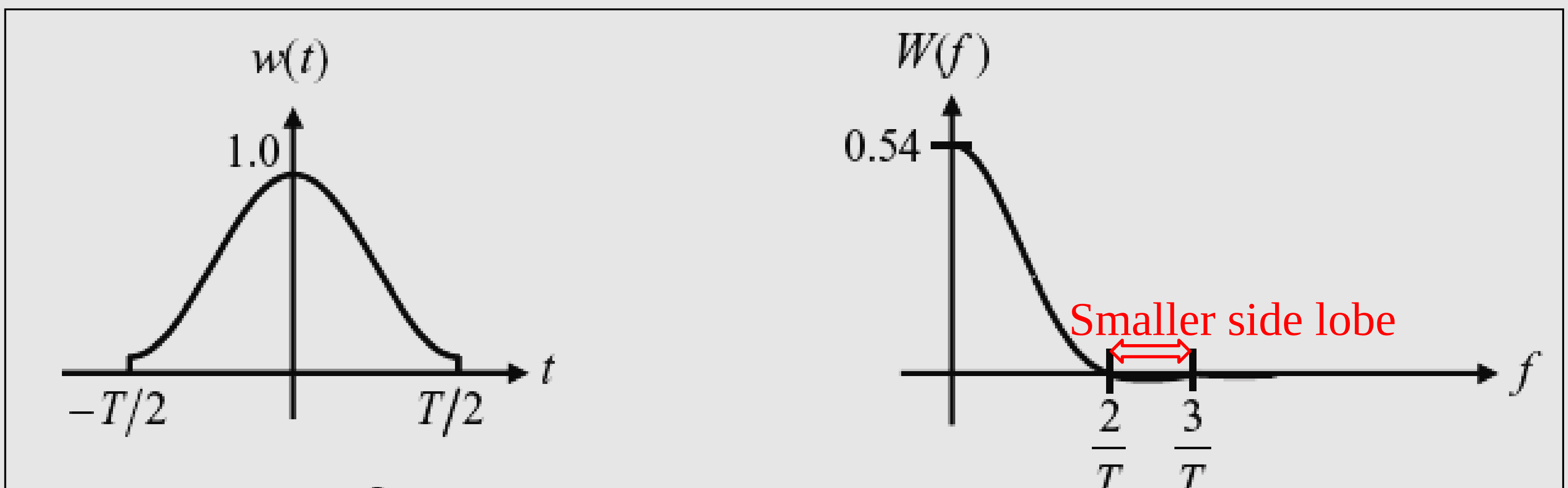


Common window functions

Rectangular Window



Hann Window



Some comments

- The rectangular window may be good for separating closely spaced sinusoidal components, but the leakage is the price to pay.

- The Hann window is a good general purpose window, and has a moderate frequency resolution and a good side lobe roll-off characteristic.

Discrete Fourier transform

- Consider a sequence x(nΔ) at n = 0, 1, 2, 3, 4, .., N-1 points. The DFT is defined as :

$$X(e^{j2\pi f\Delta}) = \sum_{n=0}^{N-1} x(n\Delta)e^{-j2\pi fn\Delta}$$

- Note that this is still continuous in frequency

Discrete Fourier transform

Now let us evaluate this at frequencies: $f = k/N\Delta$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk}$$

$$X(k) = \left[X(e^{j2\pi f\Delta}) \text{ evaluated at } f = \frac{k}{N\Delta}\text{Hz} \right] (k \text{ integer})$$

Fourier Integral vs DFT

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

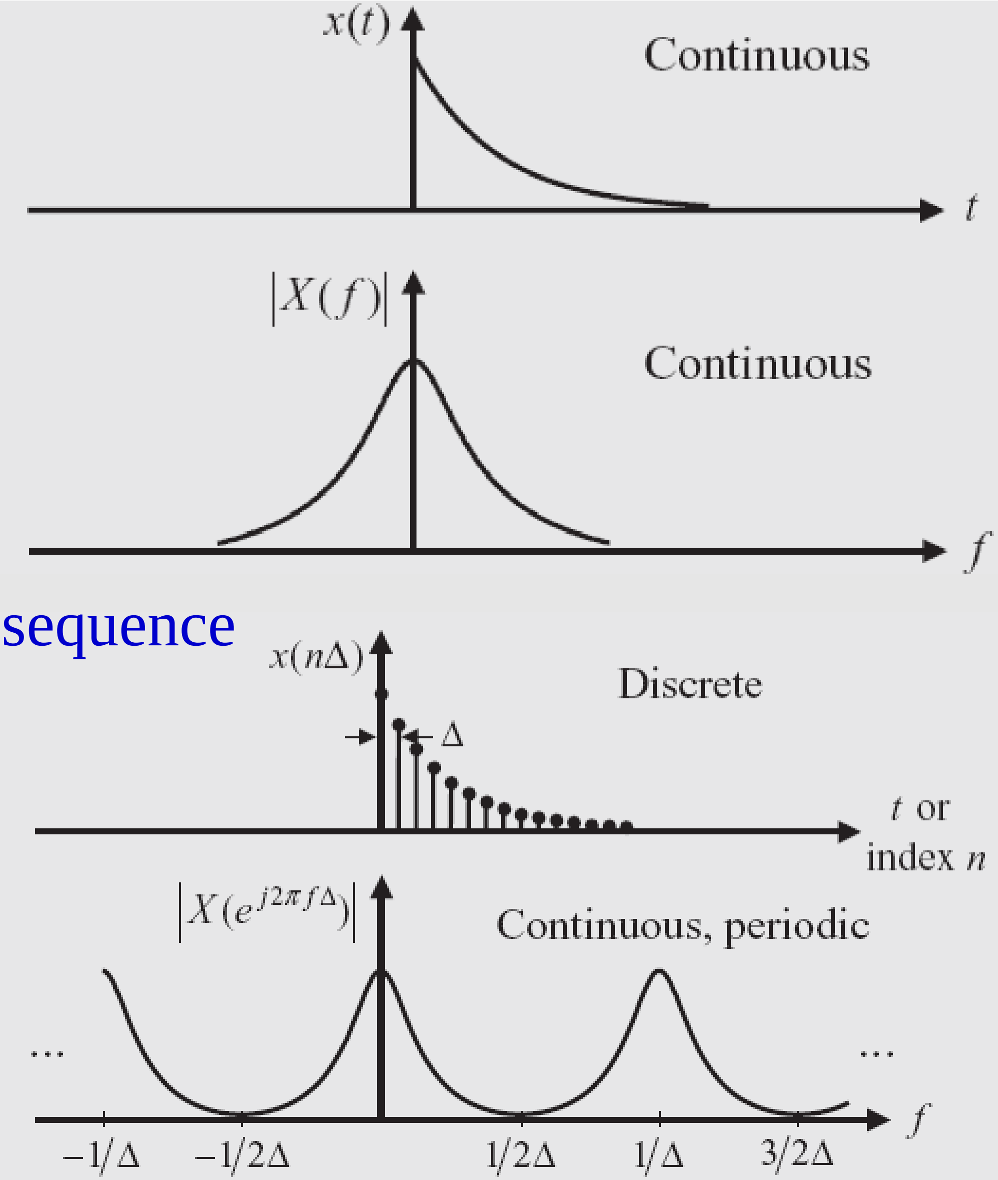
Fourier Integral

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Fourier transform of the sampled sequence

$$x(n\Delta) = \Delta \int_{-1/2\Delta}^{1/2\Delta} X(e^{j2\pi f\Delta}) e^{j2\pi fn\Delta} df$$

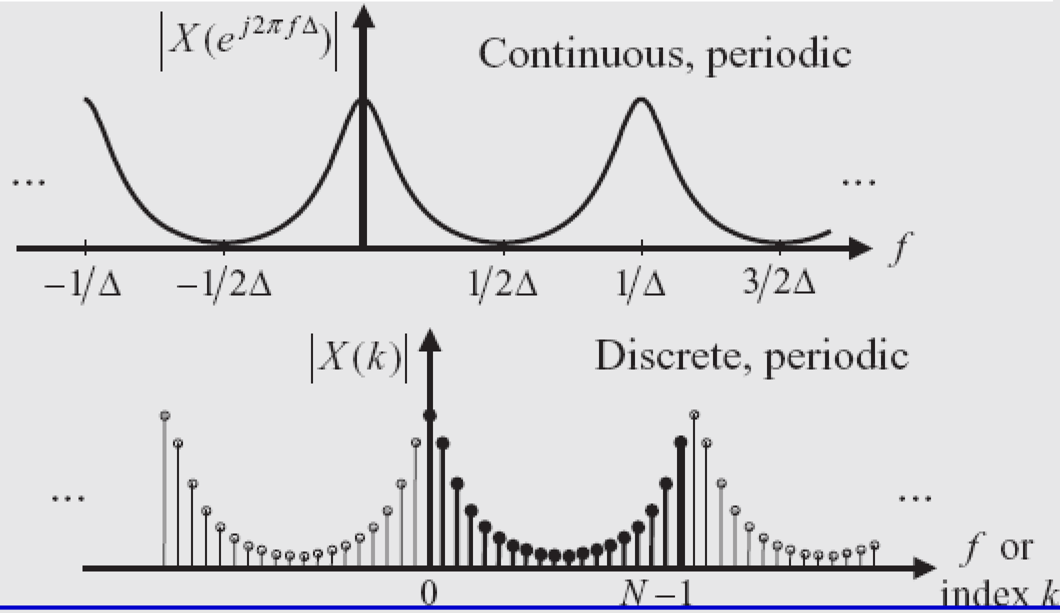
$$X(e^{j2\pi f\Delta}) = \sum_{n=-\infty}^{\infty} x(n\Delta) e^{-j2\pi fn\Delta}$$



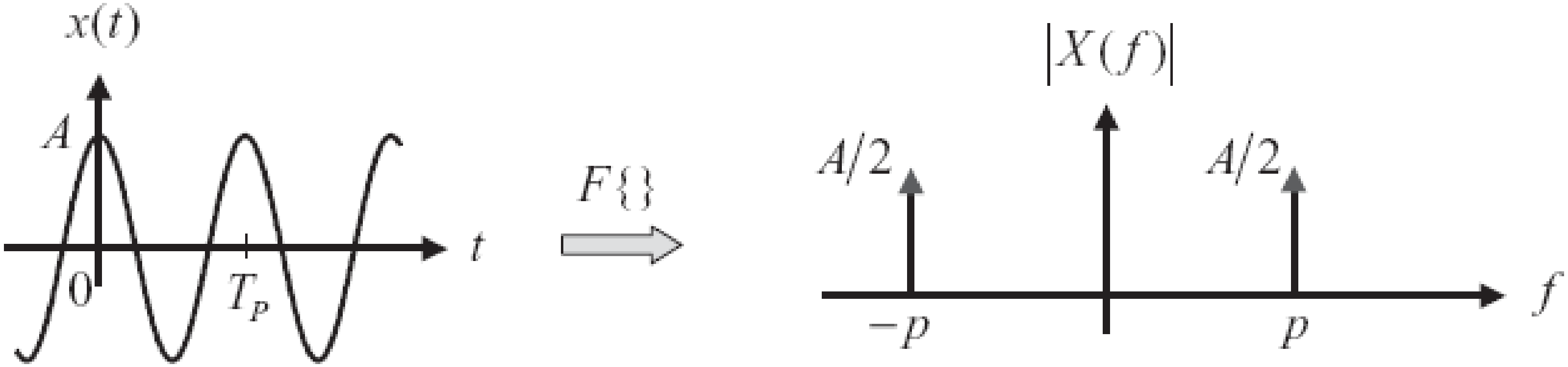
Fourier Integral vs DFT

$$X(e^{j2\pi f\Delta}) = \sum_{n=-\infty}^{\infty} x(n\Delta)e^{-j2\pi fn\Delta}$$

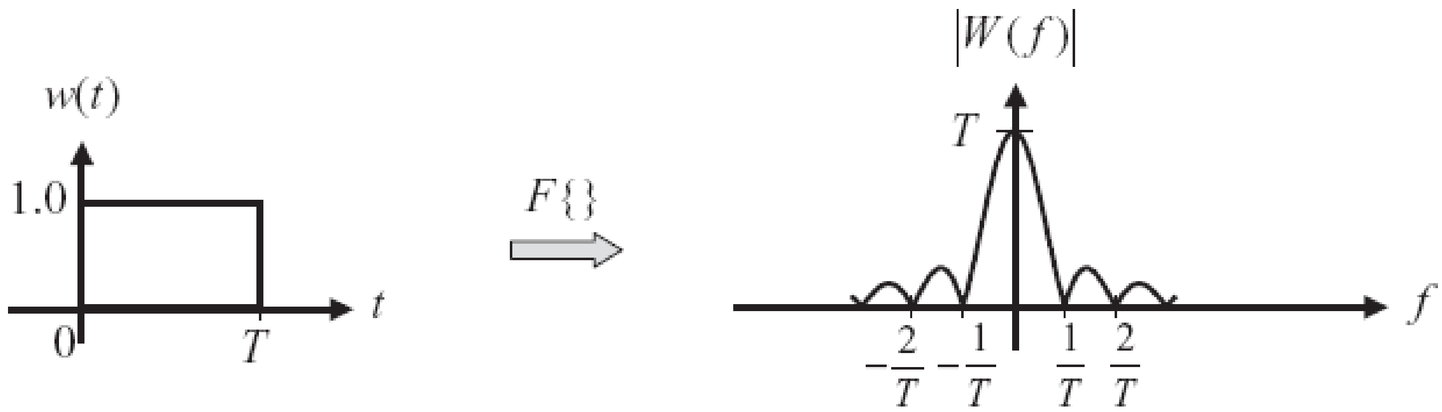
$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)nk}$$



Fourier Integral vs DFT

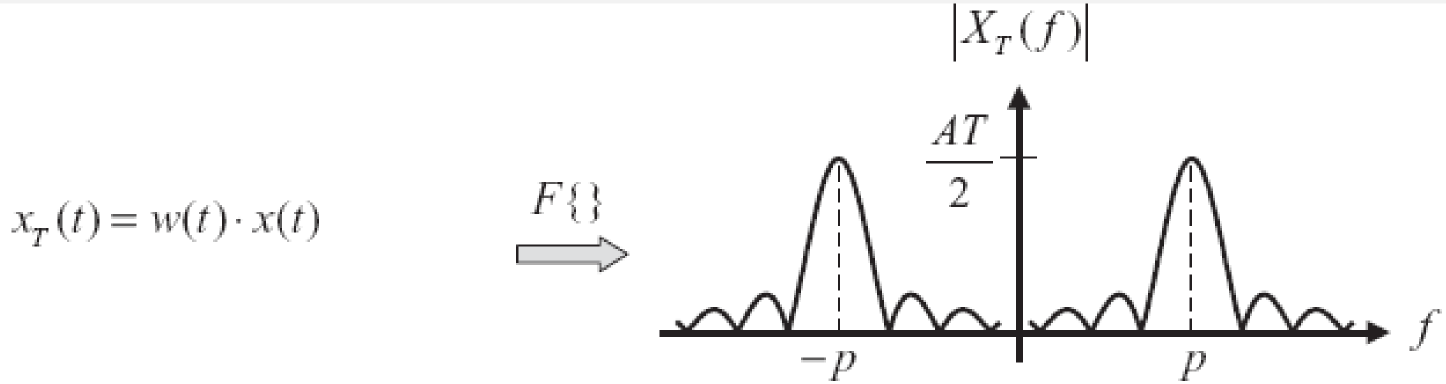


A periodic signal and its Fourier integral

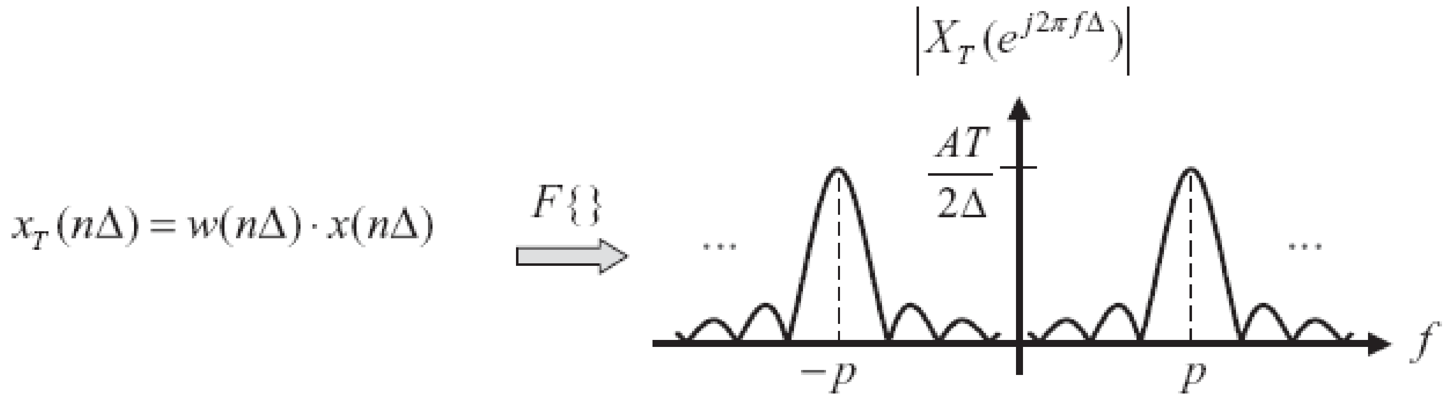


Data window and its Fourier integral

Fourier Integral vs DFT



(c) Truncated signal and its Fourier integral



(d) Truncated and sampled signal and its Fourier transform of a sequence

FFT algorithm: glimpses

- The DFT provides uniformly spaced samples of the Discrete-Time Fourier Transform (DTFT)

- DFT definition:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}} \qquad x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k] e^{j\frac{2\pi nk}{N}}$$

- Requires N^2 complex multiplications & N(N-1) complex additions

FFT algorithm: glimpses

- Take advantage of the symmetry and periodicity of the complex exponential (let $W_N = e^{-j2p / N}$)
  - symmetry: $W_N^{k[N-n]} = W_N^{-kn} = (W_N^{kn})^*$
  - periodicity: $W_N^{kn} = W_N^{k[n+N]} = W_N^{[k+N]n}$
- Note that two N/2 DFTs take less computation than one length N DFT: $2(N/2)^2 < N^2$