CE 513: Statistical Methods in Civil Engineering

Lecture- 11: Random process

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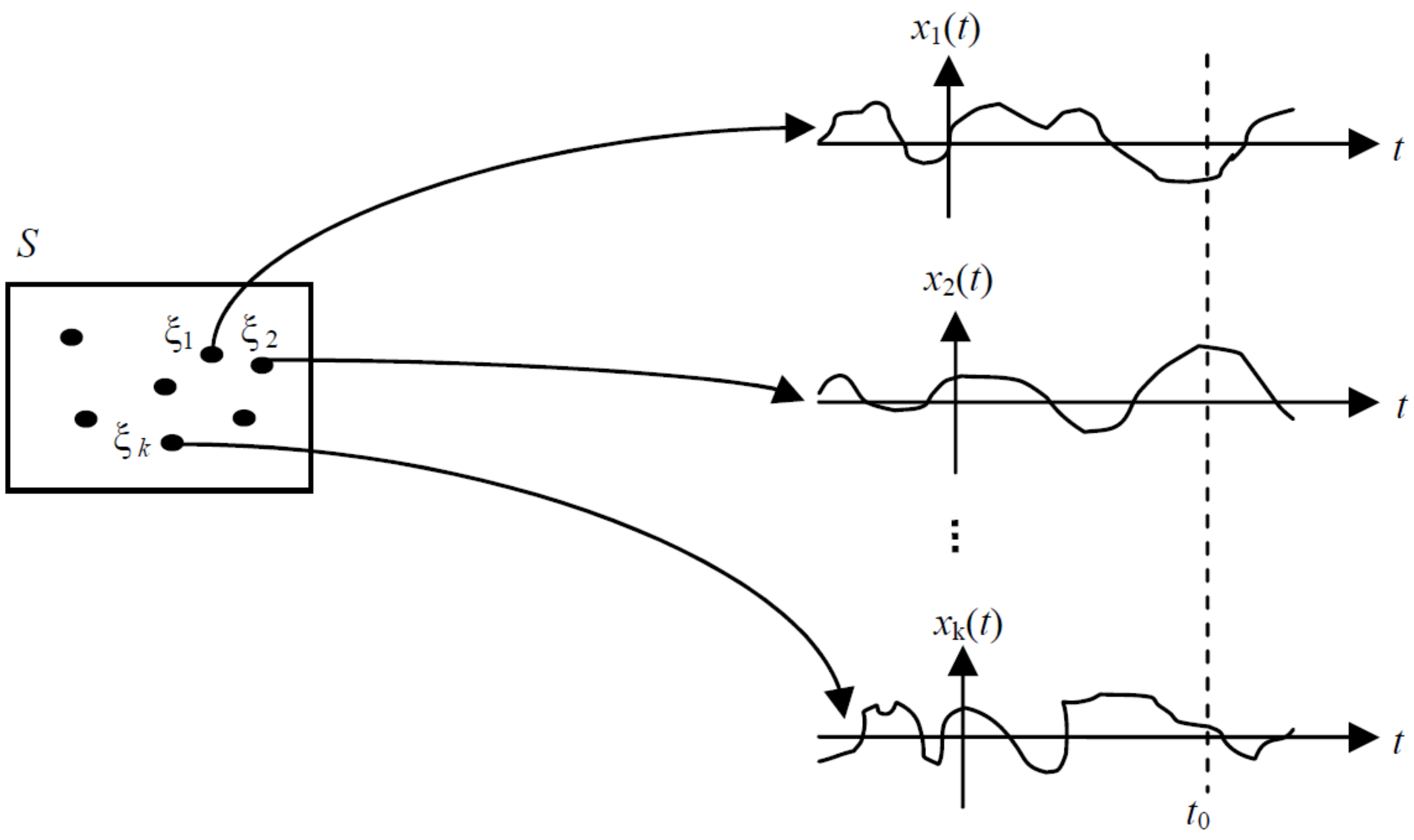
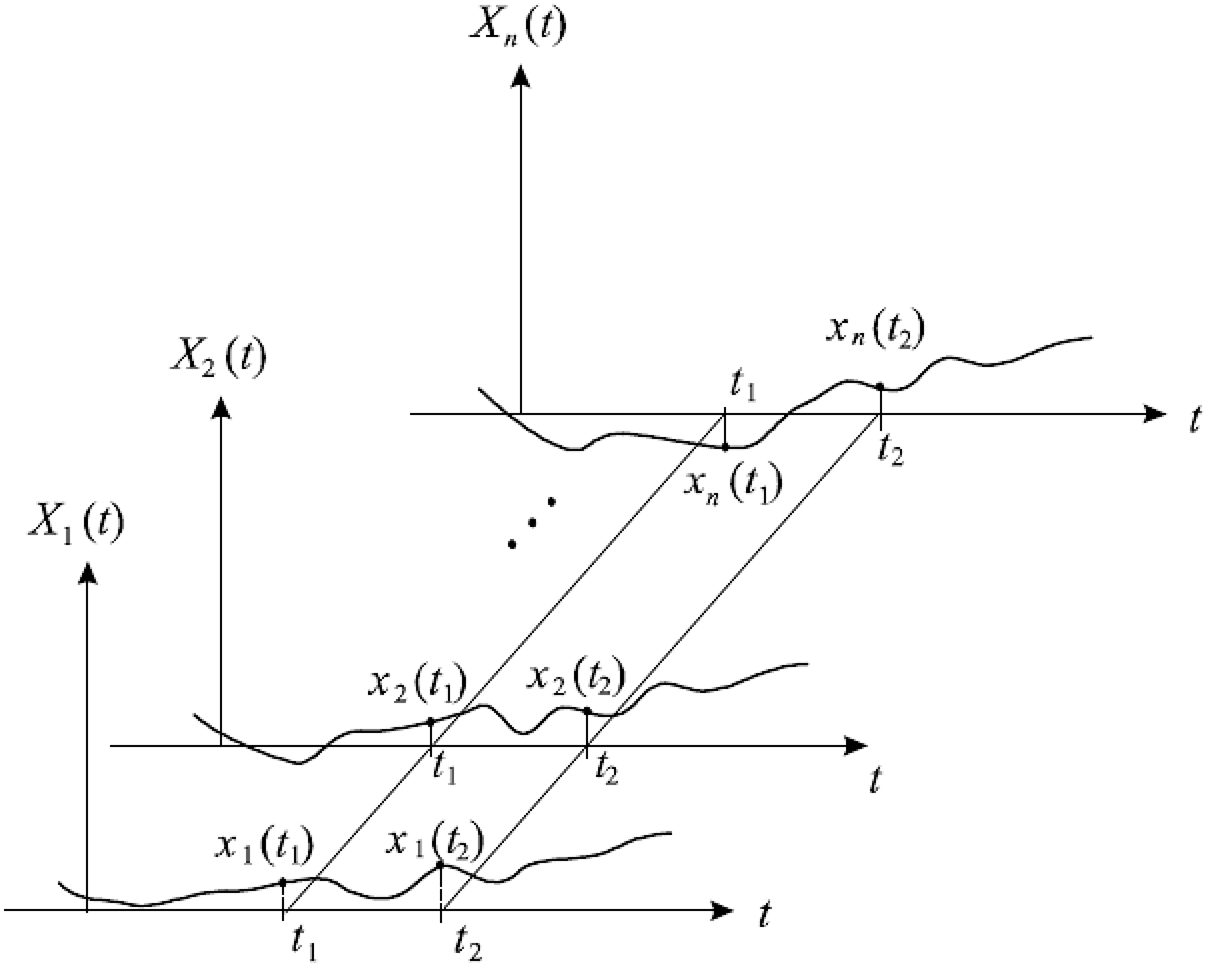
Random process



Random process

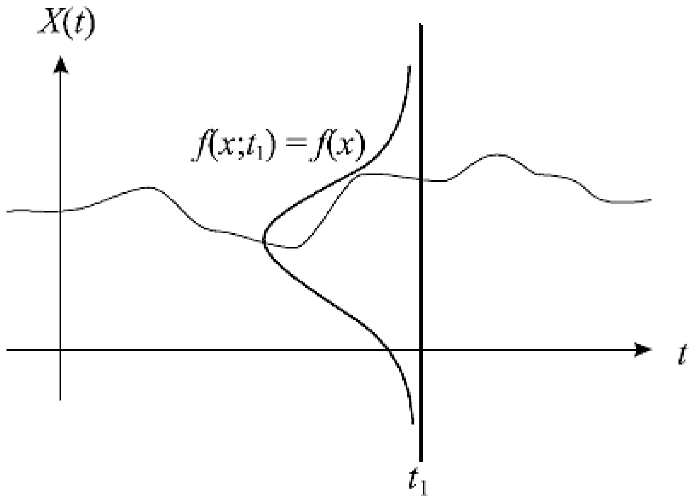


Random process

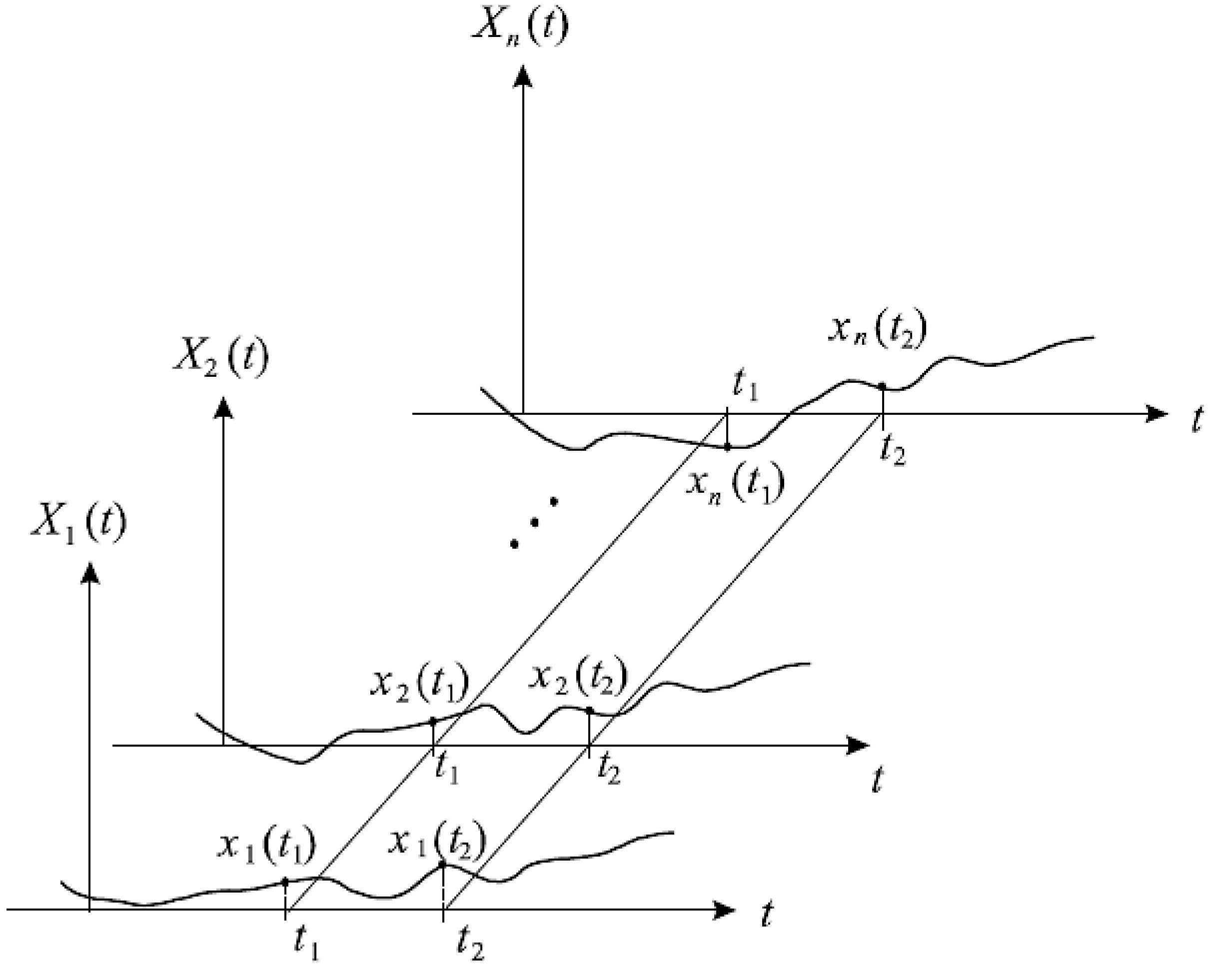
First-order distribution (for a particular value of t)

$$F_X(x;t) = P\left[X(t_0) \le x\right]$$

First-order density function $f_X(x;t) = \frac{d}{dx} F_X(x;t)$



2nd Order Averages



2nd Order Averages

2nd order distribution

$$F_X(x_1, x_2; t_1, t_2) = P[X(t_1) \le x_1 \text{ and } X(t_2) \le x_2]$$

2nd order density function

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2; t_1, t_2)$$

Expectations

Ensemble Average

The *mean* of $X(t)$ is defined by

$$\mu_X(t) = E[X(t)]$$

$X(t)$ is treated as a random variable for a fixed value of t

Autocorrelation

$$R_{xx}(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_{X_1X_2}(x_1,x_2;t_1,t_2)\,dx_1 dx_2$$

Autocovariance

$$K_X(t,s) = \mathrm{Cov}[X(t),X(s)] = E\{[X(t)-\mu_X(t)][X(s)-\mu_X(s)]\}$$
$$= R_X(t,s) - \mu_X(t)\mu_X(s)$$

Random process

The random process $X(t)$ is given by

$$X(t) = A\cos(\omega t - \Phi),$$

where A and Φ are random variables with the probability density function,

$$f_{A\Phi}(a,\phi) = \frac{1}{2\pi}\left(1 + (3a-1)\cos\phi\right),$$
$$\text{for} \quad 0 \le \phi \le 2\pi$$
$$\text{and} \quad 0 \le a \le 1.$$

Derive (a) μ_X, (b) σ_X^2, and (c) $R_{XX}(t_1, t_2)$.

Random process

(a) The mean can be found by taking the expected value of $X(t)$ or

$$E\{X(t)\} = E\{A\cos(\omega t - \Phi)\},$$

which can be expanded to

$$E\{X(t)\} = E\{A(\cos\omega t\cos\Phi + \sin\omega t\sin\Phi)\}.$$

Since only A and ϕ are random, $\cos\omega t$ and $\sin\omega t$ can be taken out of the expectation so that

$$E\{X(t)\} = \cos\omega t E\{A\cos\Phi\} + \sin\omega t E\{A\sin\Phi\},$$

Random process

$$E\{X(t)\} = \cos\omega t E\{A\cos\Phi\} + \sin\omega t E\{A\sin\Phi\},$$

where

$$\begin{aligned} E\{A\cos\Phi\} &= \int_0^1\int_0^{2\pi} a\cos\phi f_{A\Phi}(a,\phi)\,d\phi da \\ &= \frac{1}{4} \\ E\{A\sin\Phi\} &= \int_0^1\int_0^{2\pi} a\sin\phi f_{A\Phi}(a,\phi)\,d\phi da \\ &= 0. \end{aligned}$$

Then,

$$E\{X(t)\} = \frac{1}{4}\cos\omega t,$$

which means that $X(t)$ is a nonstationary random process.

Random process

(b) The variance can be found using

$$\sigma_X^2 = E\left\{(X(t) - \mu_X)^2\right\} = E\left\{X^2\right\} - \mu_X^2.$$

The root mean square $E\left\{X^2\right\}$ is given by

$$\begin{aligned}
E\left\{X^2\right\} &= E\left\{A^2(\cos\omega t\cos\Phi + \sin\omega t\sin\Phi)^2\right\} \\
&= E\{A^2\cos^2\omega t\cos^2\Phi + A^2\sin^2\omega t\sin^2\Phi \\
&\quad +2A^2\cos\omega t\cos\Phi\sin\omega t\sin\Phi\} \\
&= \cos^2\omega t E\left\{A^2\cos^2\Phi\right\} + \sin^2\omega t E\left\{A^2\sin^2\Phi\right\} \\
&\quad +2\cos\omega t\sin\omega t E\left\{A^2\cos\Phi\sin\Phi\right\}.
\end{aligned}$$

Random process

(b) The variance can be found

Each term in the previous equation can be evaluated as follows

$$\begin{aligned} E\left\{A^2 \cos^2 \Phi\right\} &= \frac{1}{6} \\ E\left\{A^2 \sin^2 \Phi\right\} &= \frac{1}{6} \\ E\left\{A^2 \cos \Phi \sin \Phi\right\} &= 0. \end{aligned}$$

Then,

$$E\left\{X^2\right\} = \frac{1}{6}$$

and the variance equals

$$\begin{aligned} \sigma_X^2 &= R_{XX}(t,t) - \mu_X^2 \\ &= E\left\{X^2\right\} - \mu_X^2 = \frac{1}{6} - \frac{1}{16} \cos \omega^2 t. \end{aligned}$$

Random process

(c) The autocorrelation function $R_{XX}(t_1, t_2)$, by definition, is given by

$$\begin{aligned} R_{XX}(t_1, t_2) &= E\left\{A^2 \cos(\omega t_1 - \Phi)\cos(\omega t_2 - \Phi)\right\} \\ &= \cos\omega t_1 \cos\omega t_2 E\{A^2\cos^2\Phi\} + \sin\omega(t_1 + t_2) E\left\{A^2 \cos\Phi \sin\Phi\right\} \\ &\quad + \sin\omega t_1 \sin\omega t_2 E\{A^2 \sin^2\Phi\} \\ &= \frac{1}{6}\cos\omega(t_1 - t_2). \end{aligned}$$

Autocorrelation: example

Consider the random process $X(t)$ $\quad X(t) = Y \cos \omega t \qquad t \geq 0$

where ω is a constant and Y is a uniform r.v. over $(0, 1)$.

(a) Find $E[X(t)]$.

(b) Find the autocorrelation function $R_x(t, s)$ of $X(t)$.

(c) Find the autocovariance function $K_x(t, s)$ of $X(t)$

Autocorrelation: example

(a) $E(Y) = \frac{1}{2}$ and $E(Y^2) = \frac{1}{3}$. Thus,

$$E[X(t)] = E(Y\cos\omega t) = E(Y)\cos\omega t = \frac{1}{2}\cos\omega t$$

(b)
$$R_X(t,s) = E[X(t)X(s)] = E(Y^2\cos\omega t\cos\omega s)$$
$$= E(Y^2)\cos\omega t\cos\omega s = \frac{1}{3}\cos\omega t\cos\omega s$$

(c)
$$K_X(t,s) = R_X(t,s) - E[X(t)]E[X(s)]$$
$$= \frac{1}{3}\cos\omega t\cos\omega s - \frac{1}{4}\cos\omega t\cos\omega s$$
$$= \frac{1}{12}\cos\omega t\cos\omega s$$

Classification of stochastic process

Strictly stationary

A random process $\{X(t), t \in T\}$ is said to be *stationary* or *strict-sense stationary* if, for all n and for every set of time instants $(t_i \in T, i = 1, 2, \ldots, n\}$,

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1 + \tau, \ldots, t_n + \tau)$$

Thus both first order and second order distributions are independent of $\boldsymbol{\tau}$

$$\boxed{\begin{aligned} F_X(x; t) &= F_X(x; t+\tau) = F_X(x) \\ f_X(x; t) &= f_X(x) \\ \mu_X(t) &= E[X(t)] = \mu \\ \mathrm{Var}[X(t)] &= \sigma^2 \end{aligned}}$$

$$\boxed{\begin{aligned} F_X(x_1, x_2; t_1, t_2) &= F_X(x_1, x_2; t_2 - t_1) \\ f_X(x_1, x_2; t_1, t_2) &= f_X(x_1, x_2; t_2 - t_1) \end{aligned}}$$

Wide sense stationary

If stationary condition of a random process X(t) does not hold for all n but holds for n ≤ k, then we say that the process X(t)is stationary to order k.

If X(t) is stationary to order 2, then X(t) is said to be wide-sense stationary (WSS) or weak stationary.

1. $E[X(t)] = \mu$ (constant)
2. $R_X(t, s) = E[X(t)X(s)] = R_X(|s - t|)$

Wide sense stationary

- Stationarity of a random process
  is analogous to
  steady state in vibration problems

- One or more of the properties of random process becomes independent of time

- Strong sense stationarity (SSS) : defined with respect to pdf-s

- Wide sense stationarity (WSS) : defined with respect to moments

Stationary SS: Few Theorems

1. If a random process which is stationary to order **n** is also stationary to all orders lower than **n.**

2. If $\{X(t), t \in T\}$ is a strict-sense stationary random process, then it is also WSS.

3. If a random process X(t) is WSS, then it must also be covariance stationary

SSS: Example

Consider a random process $X(t)$ defined by

$$X(t) = U \cos \omega t + V \sin \omega t \qquad -\infty < t < \infty$$

where ω is constant and U and V are r.v.'s.

(*a*) Show that the condition

$$E(U) = E(V) = 0$$

is necessary for $X(t)$ to be stationary.

(*b*) Show that $X(t)$ is WSS if and only if U and V are uncorrelated with equal variance; that is,

$$E(UV) = 0 \qquad E(U^2) = E(V^2) = \sigma^2$$

(*a*)

$$\mu_X(t) = E[X(t)] = E(U)\cos\omega t + E(V)\sin\omega t$$

must be independent of t for $X(t)$ to be stationary.

This is possible only if $\mu_x(t) = 0$, that is, $E(U) = E(V) = 0$.

(*b*) If $X(t)$ is WSS, then

$$E[X^2(0)] = E\left[X^2\left(\frac{\pi}{2\omega}\right)\right] = R_{XX}(0) = \sigma_X{}^2$$

But $X(0) = U$ and $X(\pi/2\omega) = V$; thus,

$$E(U^2) = E(V^2) = \sigma_X{}^2 = \sigma^2$$

Using the above result, we obtain

$$\begin{aligned} R_x(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= E\{(U\cos\omega t + V\sin\omega t)[U\cos\omega(t+\tau) + V\sin\omega(t+\tau)]\} \\ &= \sigma^2\cos\omega\tau + E(UV)\sin(2\omega t + \omega\tau) \end{aligned}$$

Conversely, if $E(UV) = 0$ and $E(U^2) = E(V^2) = \sigma^2$, then from the result of part (*a*) and the above result

$$\mu_x(t) = 0$$
$$R_x(t, t+\tau) = \sigma^2\cos\omega\tau = R_x(\tau)$$

Autocorrelation: Properties

1. It is an even function of τ

$$R_x(\tau) = R_x(-\tau)$$

2. Bounded by its value at origin

$$|R_x(\tau)| \leq R_x(0)$$

3. $R_x(0) = E[X^2]$

4. If X is periodic $R_x(\tau)$ is also periodic

$R_x(\tau)$ (WSS) examples

1) $G(t) = A\cos(\omega_0 t + \phi)$, where ϕ is uniform RV with $\phi \sim U(0, 2\pi)$. Determine the mean and the autocorrelation ?

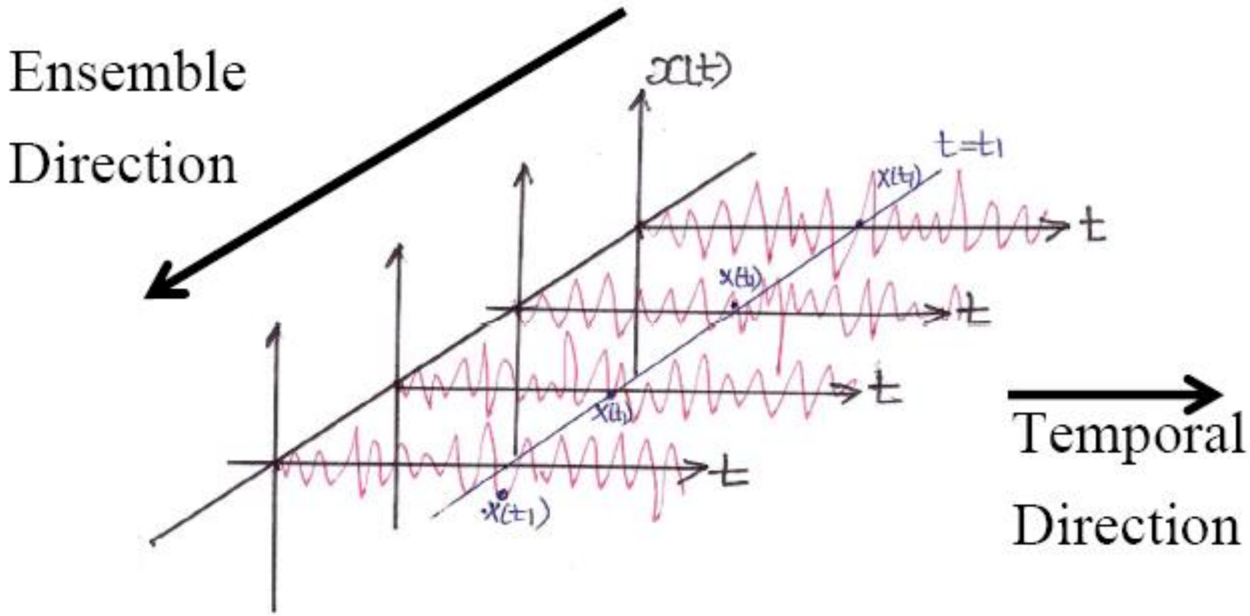
Ans = $\frac{A^2}{2}\cos(\omega_0 \tau)$

2) $G(t) = A\cos(\omega t + \theta)$, where ω and θ are independent RVs with $\theta \sim U(0, 2\pi)$ and $\omega \sim U(\omega_1, \omega_2)$. Determine the mean and the autocorrelation ?

Ans = $\frac{A^2}{2\tau(\omega_2 - \omega_1)}[\sin\omega_2\tau - \sin\omega_1\tau]$

Erogodicity

Basic idea: Equivalence of temporal and ensemble averages



Erogodicity

A random process is said to be Ergodic if it has the property that the time averages of sample functions of the process are equal to the corresponding statistical or ensemble averages.

$$E[X(t)] = \langle X(t) \rangle = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt$$

The sample autocorrelation can be calculated using the following formula

$$R_X(\tau) = \langle X(t)X(t+\tau) \rangle = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt$$

Erogodicity

- Consider a sample of a random process: x (1), x (2),………x (N)
- The sample mean of the sequence could be estimated as:

$$\widehat{m_x}(N) = \frac{1}{N}\sum_{n=0}^{N-1} x_n$$

- Since the sample is a realization of a random process it must have a constant ensemble mean E[X(n)]= m_x

If the sample mean $\widehat{m_x}(N)$ of a WSS converges to m_x in a mean square sense as N→ ∞, then the random process is said to be Ergodic in mean

$$\lim_{N\to\infty} \widehat{m_x}(N) = m_x$$

Mean Ergodic Theorem

Mean Ergodic Theorem 1. Let $x(n)$ be a WSS random process with autocovariance sequence $c_x(k)$. A necessary and sufficient condition for $x(n)$ to be ergodic in the mean is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} c_x(k) = 0$$

Mean Ergodic Theorem 2. Let $x(n)$ be a WSS random process with autocovariance sequence $c_x(k)$. Sufficient conditions for $x(n)$ to be ergodic in the mean are that $c_x(0) < \infty$ and

$$\lim_{k\to\infty} c_x(k) = 0$$

Sample autocorrelation of a WSS and Ergodic process

$$r_x(k) = E[\, x(k)x(n-k)]$$

For each k, the autocorrelation is the expected value of the process: $y_k(n) = x(k)x(n-k)$

Using Ergodicity properties, the autocorrelation is finally estimated as :

$$\widehat{r_x}(k, N) = \frac{1}{N} \sum_{n=0}^{N-1} x(k)x(n-k)$$

WSS& Ergodic process: example

Coming back to the random phase sinusoid

$G(t) = A\cos(\omega_0 t + \phi)$, where ϕ is uniform RV with $\phi \sim U(0, 2\pi)$.

$$\langle X(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt \;= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\cos(\omega_0 t + \phi)\, dt = 0$$

$$\langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A^2\cos(\omega_0 t + \omega_0\tau + \phi)cos(\omega_0 t + \phi)dt$$

$$= \frac{A^2}{2}\cos(\omega_0\tau)$$

FOURIER TRANSFORM

- Extension of Fourier analysis to non-periodic phenomena
- Discrete to continuous
- Skipping essential steps, in the limit $T_p \rightarrow \infty$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

Inverse
Fourier transform

POWER SPECTRUM

Knowing that a random process is composed of energy at many frequencies, we define a random process that is a sum of harmonics, similarly to the Fourier series. Begin first with a random function of one harmonic process, $X(t) = C\cos(\omega t - \phi)$, or equivalently,

$$X(t) = A\cos\omega t + B\sin\omega t,$$

where A and B are independent random variables. Assuming both to be zero mean and identically distributed, we have

$$\begin{aligned} \mu_A = \ & \mu_B = \ 0 \\ \sigma_A^2 = \ & \sigma_B^2 = \ \sigma^2. \end{aligned}$$

POWER SPECTRUM

The autocorrelation is given by

$$\begin{aligned} R_{XX}(\tau) &= E\{X(t)\,X(t+\tau)\} \\ &= E\{(A\cos\omega t + B\sin\omega t)\,(A\cos\omega\,[t+\tau] + B\sin\omega\,[t+\tau])\}\,. \end{aligned}$$

Expanding the product and utilizing trigonometric identities results in

$$R_{XX}(\tau) = \sigma^2 \cos\omega\tau.$$

Suppose that the frequency content of the random process is expanded,

$$\begin{aligned} X(t) &= \sum_{k=1}^{m} X_k(t) \\ &= \sum_{k=1}^{m} (A_k \cos\omega_k t + B_k \sin\omega_k t)\,, \end{aligned}$$

where we make the same assumptions as were made above about A and B.

POWER SPECTRUM

Suppose that the frequency content of the random process is expanded,

$$X(t) = \sum_{k=1}^{m} X_k(t) = \sum_{k=1}^{m} \left(A_k \cos\omega_k t + B_k \sin\omega_k t\right),$$

where we make the same assumptions as were made above about A and B. Following the same procedure as for the above single-frequency process, we find

$$R_{XX}(\tau) = \sum_{k=1}^{m} R_{X_k X_k}(\tau) = \sum_{k=1}^{m} \sigma_k^2 \cos\omega_k \tau.$$

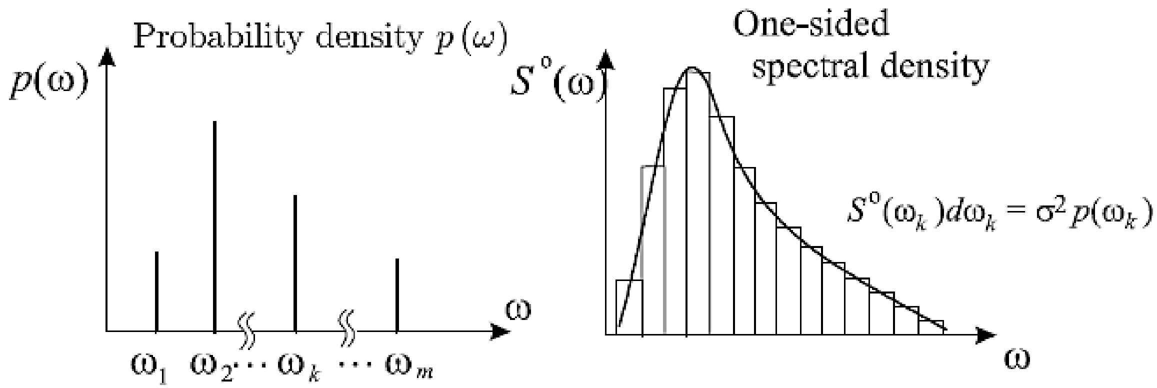
The total variance for the process is found by recalling that

$$\sigma^2 = E\left\{X^2(t)\right\} - \mu_X^2 = R_{XX}(0),$$

the last equality being true for the case where the mean equals zero.

$$\sigma^2 = \sum_{k=1}^{m} \sigma_k^2.$$

POWER SPECTRUM



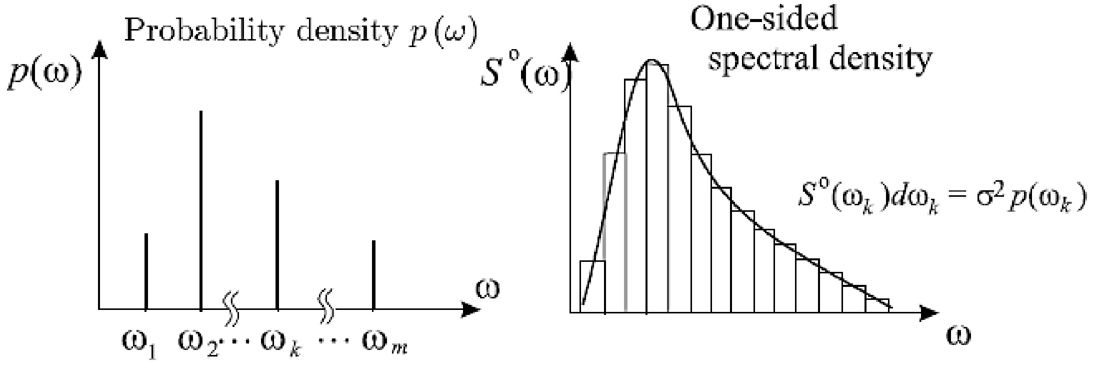
(a) (b)

Each frequency component ω_k contributes σ_k^2 to the total variance σ^2. The fraction of the total is given by the ratio σ_k^2/σ^2, which can be defined as $p(\omega_k)$ as shown in Figure Note that $\sum_{k=1}^{m} p(\omega_k) = 1$. Then,

$$R_{XX}(\tau) = \sigma^2 \sum_{k=1}^{m} p(\omega_k) \cos \omega_k \tau,$$

where $p(\omega_k)$ acts as a weighting function. The above implies that $p(\omega_k)$ behaves like a probability density.

POWER SPECTRUM



(a) (b)

Suppose the frequency spectrum becomes very broad, including many frequencies, that is $m \to \infty$, resulting in a continuous frequency spectrum. Define $d\omega = \omega_{k+1} - \omega_k$. In an analogous manner to how we proceeded from a discrete to a continuous probability density function, replace $\sigma^2 p(\omega_k)$ by $S^o(\omega)\,d\omega$, and the sum above by an integral over the frequency range,

$$R_{XX}(\tau) = \int_0^\infty S^o(\omega) \cos \omega\tau \, d\omega.$$

$S^o(\omega)$ is called the *one-sided spectral density* of the random process because it distributes the variance of the random process as a density across the frequency spectrum. The one-sided spectral density is shown in Figure

WIENER KHINCHINE THEOREM

$$S_{XX}(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\omega\tau}\, d\omega.$$

PROPERTIES

Since $R_{XX}(\tau) = R_{XX}(-\tau)$, $S_{XX}(\omega)$ is not a complex function but a real even function,

$$S_{XX}(\omega) = S_{XX}(-\omega).$$

For $\tau = 0$,

$$\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = R_{XX}(0) = E\{X^2(t)\} \geq 0.$$

On physical grounds, then, it can be argued that since the area under the power spectral density equals σ_X^2 for a zero-mean random process, it must be a positive quantity for any $\Delta\omega$, that is, $S_{XX}(\omega) \geq 0$.

PROPERTIES

For $\tau = 0$,

$$\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = R_{XX}(0) = E\{X^2(t)\} \geq 0.$$

The above integral represents average or, mean-square power of the process X(t)

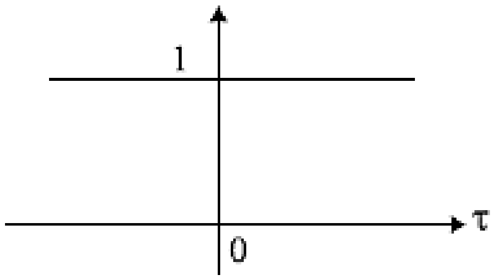
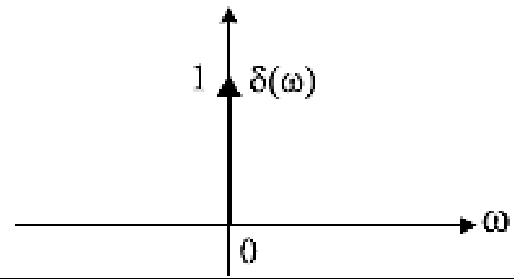
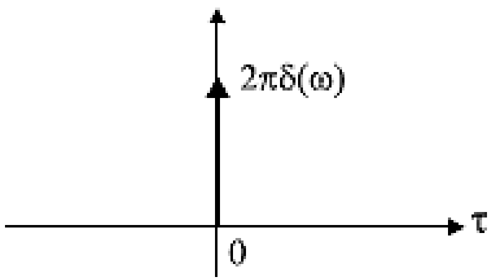
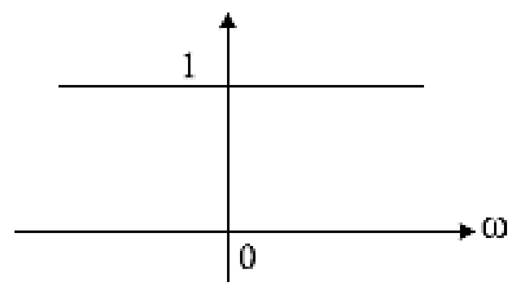
For an ergodic process, the expected value can be written as

$$E\{X^2(t)\} \simeq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X^2(t)dt,$$

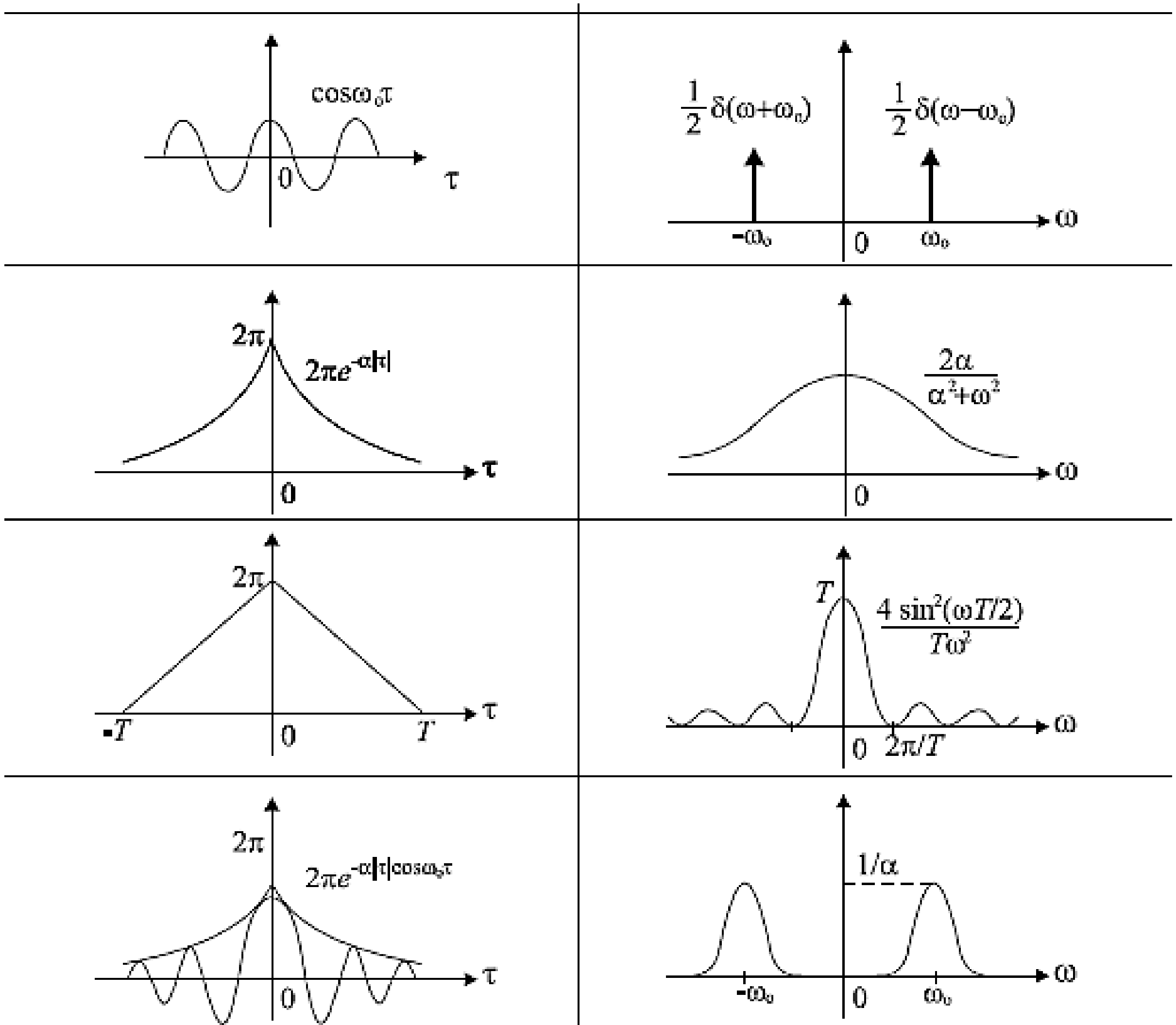
Which is the total energy over the total time or the average power.

Therefore, power spectrum is a measure of the energy

Certain important $R_x(\tau) \leftrightarrow S_x(\omega)$

$R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$	$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$
	
	

Certain important $R_x(\tau) \leftrightarrow S_x(\omega)$



Example

Consider for example $R_x(\tau) = \sin \omega_0 \tau$

The spectral density is related to the autocorrelation function by Equation

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau.$$

If the autocorrelation is a pure sine function, $S_{XX}(\omega)$ is given by the integral,

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sin \omega_o \tau \, e^{-i\omega\tau} d\tau.$$

Example

Consider for example $R_x(\tau) = \sin\omega_0\tau$

Using the Euler identity, $\sin\omega_o\tau = [\exp(i\omega_o\tau) - \exp(-i\omega_o\tau)]/2i$, the spectral density becomes

$$\begin{aligned} S_{XX}(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2i}\left[\exp(i\omega_o\tau) - \exp(-i\omega_o\tau)\right]e^{-i\omega\tau}d\tau \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2i}\left(e^{-i(\omega-\omega_o)\tau} - e^{-i(\omega+\omega_o)\tau}\right)d\tau \\ &= \frac{1}{2i}\left[\delta(\omega-\omega_o) - \delta(\omega+\omega_o)\right]. \end{aligned}$$

This example shows that a sine function cannot be a valid autocorrelation function. Why ?

Units of Power spectral density

$$\text{Units of PSD: } \frac{\left[\text{Units of } X(t)\right]^2}{\text{frequency}}$$

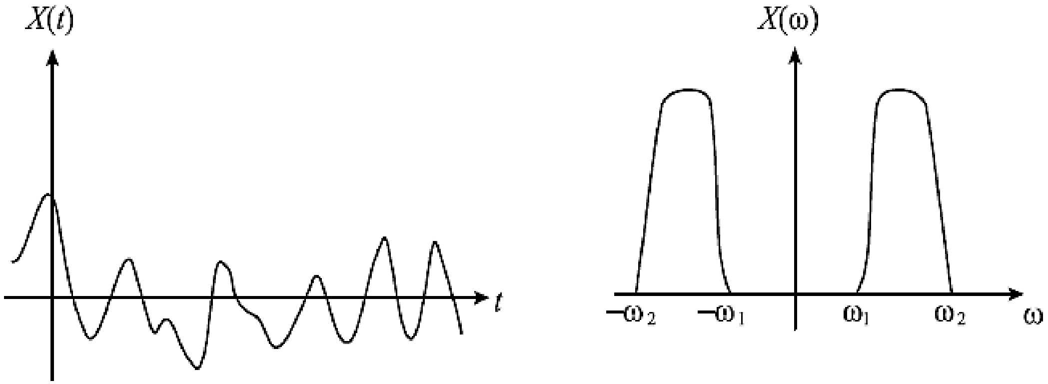
Ex: $X(t)$ is displacement

$$\text{Units of PSD : } \frac{\text{m}^2}{\text{Hz}} \text{ or } \frac{\text{m}^2}{\left(\text{rad/s}\right)}$$

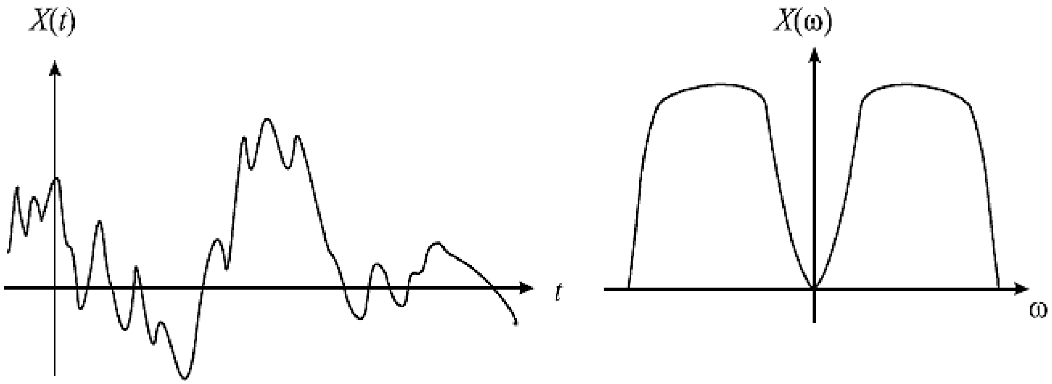
Similarly, if X(t) is acceleration

$$\text{Units of PSD: } \frac{(\text{m/s}^2)^2}{\text{Hz}} \text{ or } \frac{(\text{m/s}^2)^2}{\left(\text{rad/s}\right)}$$

Narrow band & broad band processes



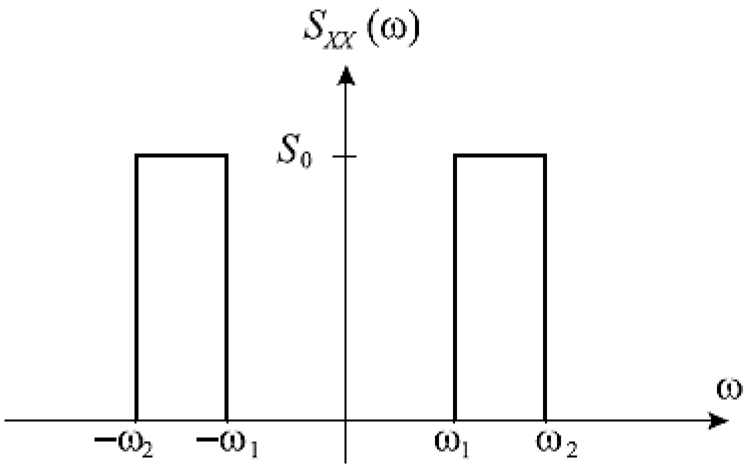
Narrow-band random process $X(t)$ in time and frequency $X(\omega)$ domains.



Broad-band random process $X(t)$ in time and frequency $X(\omega)$ domains

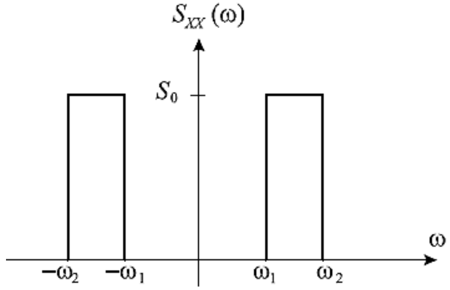
Narrow band & broad band processes

A narrow band spectrum can be expressed as a flat spectrum S_0 in the frequency band $[\omega_1\ \omega_2]$



Narrow band & broad band processes

The autocorrelation function for such a process is evaluated as follows,



$$\begin{aligned} R_{XX}(\tau) &= \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega \\ &= 2\int_{\omega_1}^{\omega_2} S_0 \cos\omega\tau \, d\omega, \end{aligned}$$

where the real part of the complex exponential is retained having made use of the symmetry of the power spectrum function. The integral is evaluated to give

$$R_{XX}(\tau) = 2\frac{S_0}{\tau}(\sin\omega_2\tau - \sin\omega_1\tau). \tag{1}$$

Note that the autocorrelation function consists of two harmonic functions at frequencies ω_1 and ω_2. When the frequencies are close to each other, beating is observed. This is clearer when Equation (1) is written as

$$R_{XX}(\tau) = 4\frac{S_0}{\tau}\cos\left\{\left(\frac{\omega_1+\omega_2}{2}\right)\tau\right\}\sin\left\{\left(\frac{\omega_2-\omega_1}{2}\right)\tau\right\}.$$

Narrow band & broad band processes

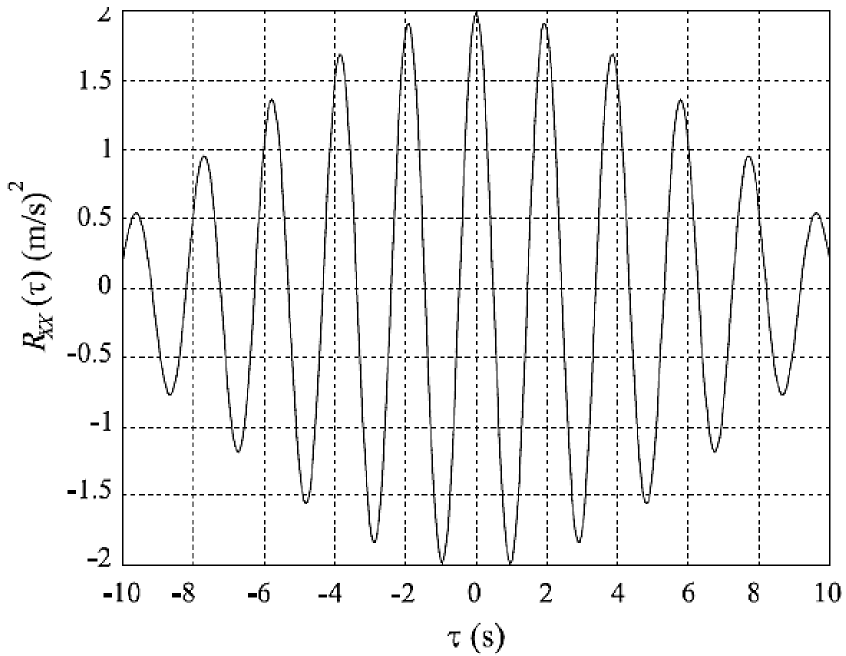


Figure : The autocorrelation function for an ideal narrow-band process. $S_o = 2\ \mathrm{m}^2/\mathrm{s}$, $\omega_1 = 3$ rad/s, and $\omega_2 = 3.5$ rad/s.

Broad band processes

A broad band process is one that contains significant energy for a wider range of frequencies

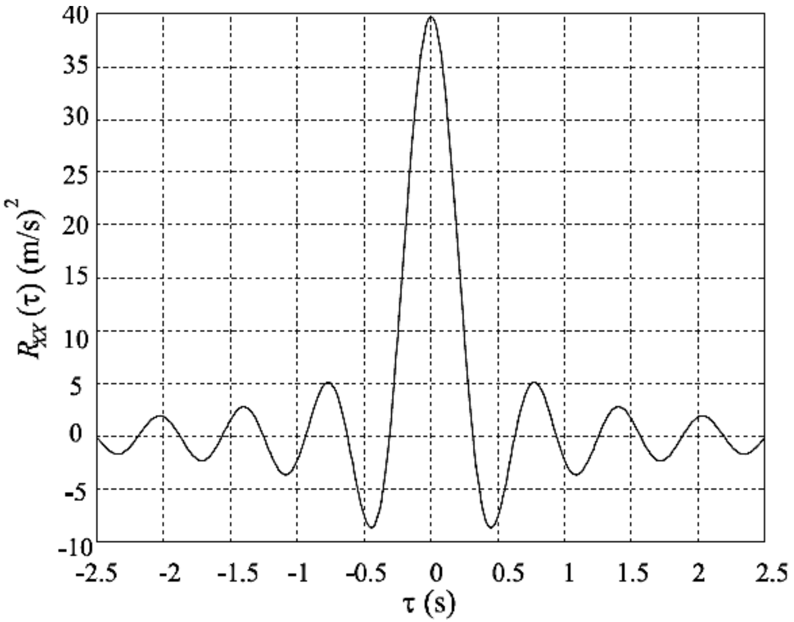


Figure : The autocorrelation function for an ideal narrow-band process $S_0 = 2$ m^2/s, $\omega_1 = 0$ rad/s, and $\omega_2 = 10$ rad/s.

White noise process

What happens when $\boldsymbol{\omega_1} = 0$ and Lim $\boldsymbol{\omega_2} \to \infty$

$$\lim_{\omega_1 \to 0} 4\frac{S_0}{\tau} \cos\left\{\left(\frac{\omega_1 + \omega_2}{2}\right)\tau\right\} \sin\left\{\left(\frac{\omega_2 - \omega_1}{2}\right)\tau\right\} = 2S_0 \frac{\sin \omega_2 \tau}{\tau}$$

$$\begin{aligned} R_{XX}(\tau) &= \lim_{\omega_2 \to \infty} 2S_0 \frac{\sin \omega_2 \tau}{\tau} \\ &= 2\pi S_0 \delta(\tau) \end{aligned}$$

Equation can be confirmed using the definition of the spectral density given in Equation

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi S_0 \delta(\tau) e^{-i\omega\tau} d\tau = S_0$$

White noise process

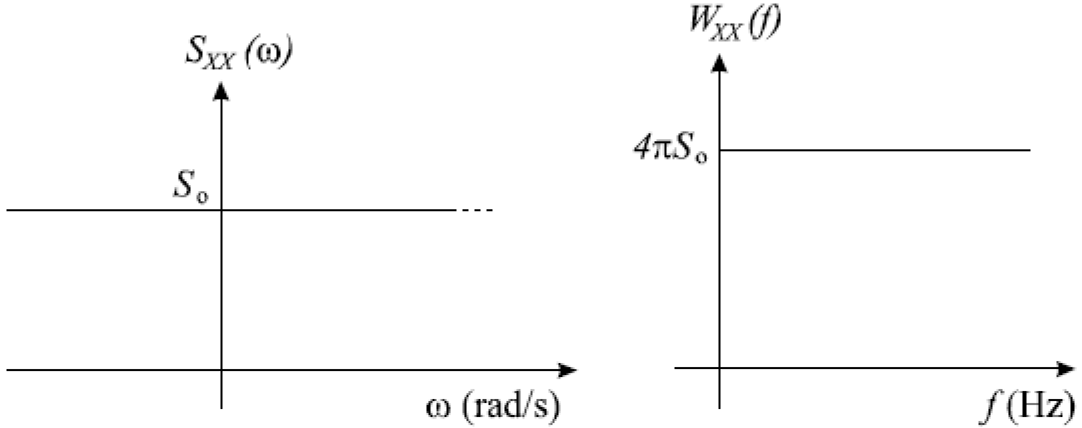


Figure : Two-sided and one-sided white noise spectra.

Fourier representation of random process

If X_t is a result of many effects that are independent or nearly Independent, then X_t is a Gaussian process according to **Central Limit Theorem**

If $X(t)$ is a stationary process on $[0, T]$, its realization can be represented using the Fourier series

$$X(t) = \sum_{n=1}^{N} a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n}{T} t,$$

where the coefficients a_n and b_n are random variables that have identical, independent normal distributions with zero mean,

$$\begin{aligned} E\{a_n\} &= E\{b_n\} = 0 \\ E\{a_n^2\} &= E\{b_n^2\} = \sigma_n^2. \end{aligned}$$

Fourier representation of random process

Independence implies

$$
\begin{aligned}
E\{a_n b_m\} &= 0, \quad \text{for } 1 \le n, m \le N \\
E\{a_n a_m\} &= 0, \quad \text{for } 1 \le n, m \le N \quad \text{and } n \ne m \\
E\{b_n b_m\} &= 0, \quad \text{for } 1 \le n, m \le N \quad \text{and } n \ne m.
\end{aligned}
$$

For example, a_1 is independent of all coefficients except for itself.

It should be noted that the number of Fourier components N should be large (at least 200) to duplicate the spectrum accurately.

Fourier representation of random process

The autocorrelation function is then given by

$$\begin{aligned} R_{XX}(\tau) &= E\{X(t)\,X(t+\tau)\} \\ &= E\left\{\left(\sum_{n=1}^{N} a_n \cos\frac{2\pi n}{T}t + b_n \sin\frac{2\pi n}{T}t\right)\right. \\ &\quad \left.\cdot\left(\sum_{m=1}^{N} a_m \cos\frac{2\pi m}{T}(t+\tau) + b_m \sin\frac{2\pi m}{T}(t+\tau)\right)\right\} \end{aligned}$$

Show that $R_{XX}(\tau)$ is given by

$$\begin{aligned} R_{XX}(\tau) &= \sum_{n=1}^{N} \sigma_n^2 \left(\cos\frac{2\pi n}{T}t\cos\frac{2\pi n}{T}(t+\tau)\right. \\ &\quad \left.+\sin\frac{2\pi n}{T}t\cos\frac{2\pi n}{T}(t+\tau)\right) \\ &= \sum_{n=1}^{N} \sigma_n^2 \cos\frac{2\pi n}{T}\tau. \end{aligned}$$

Fourier representation of random process

Show that $R_{XX}(\tau)$ is given by

$$\begin{aligned} R_{XX}(\tau) &= \sum_{n=1}^{N} \sigma_n^2 \left(\cos \frac{2\pi n}{T} t \cos \frac{2\pi n}{T}(t+\tau) \right. \\ &\qquad \left. + \sin \frac{2\pi n}{T} t \cos \frac{2\pi n}{T}(t+\tau) \right) \\ &= \sum_{n=1}^{N} \sigma_n^2 \cos \frac{2\pi n}{T} \tau . \end{aligned}$$

Hint: Since $E\{a_n b_m\} = 0$ for any n and m, we can write

$$\begin{aligned} R_{XX}(\tau) &= \sum_{n=1}^{N} \sum_{m=1}^{N} \left[E\{a_n a_m\} \cos \frac{2\pi n}{T} t \cos \frac{2\pi m}{T}(t+\tau) \right. \\ &\qquad \left. + E\{b_n b_m\} \sin \frac{2\pi n}{T} t \cos \frac{2\pi m}{T}(t+\tau) \right] . \end{aligned}$$

Using $E\{a_n a_m\} = 0$ for $n \neq m$ and $E\{a_n^2\} = E\{b_n^2\} = \sigma_n^2$ for $n = m$
the double summation can be simplified as a single summation

Fourier representation of random process

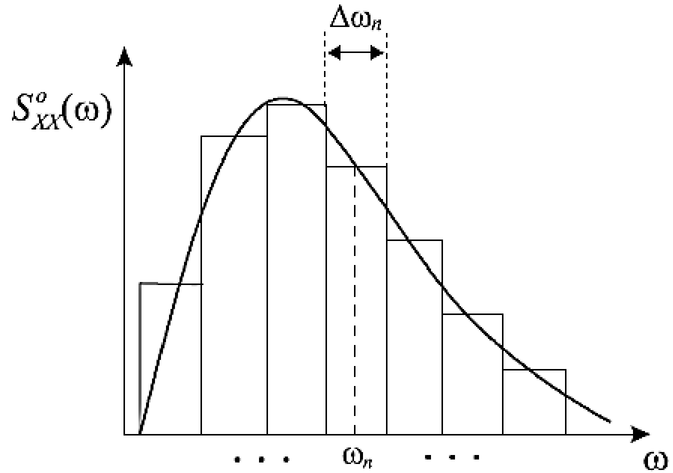


Figure : Area under the discretized spectral density curve.

The area under the spectral density is $R_x(0) = E\{X_t^2\}$

Thus, $\sigma_X^2 = \sum_{n=1}^{N} \sigma_n^2$

$$\sigma_n^2 \simeq S^o_{XX}(\omega_n)\,\Delta\omega.$$

If the functional form of $S^o_{XX}(\omega)$ is given, the integral expression can be used for accuracy. If $S^o_{XX}(\omega)$ is given as discrete data, the approximation to the integral is used. Note that $S^o_{XX}(\omega)$ is one sided in this case.

This suggests that a sample time history of the Gaussian random process $X(t)$ can be expressed as a Fourier series with independent Gaussian random numbers a_n and b_n with variance σ_n^2.

Fourier representation of random process

Another representation is given by

$$X(t) = \sum_{n=1}^{N} \sqrt{2}\sigma_n \cos\left(\frac{2\pi n}{T}t - \varphi_n\right)$$

- σ_n is as prescribed in the previous slide
- φ_n is a uniform random variable in $[0\ 2\pi]$

This representation does not satisfy the condition of a Gaussian process unless $N \to \infty$. However, there is no significant difference between two representations for a large number of Fourier components ($N > 1000$)

This was shown in Elgar S., Guza, R.T, and Seymour R.J., "Wave Group Statistics from Numerical Simulations of a Random Sea," *Applied Ocean Research*, 7(2), 93-96, 1985.

Fourier representation of random process

$$X(t) = \sum_{n=1}^{N} \sqrt{2}\sigma_n \cos\left(\frac{2\pi n}{T}t - \varphi_n\right)$$

φ_n is a uniform random variable in $[0\ 2\pi]$

The autocorrelation function is then given by

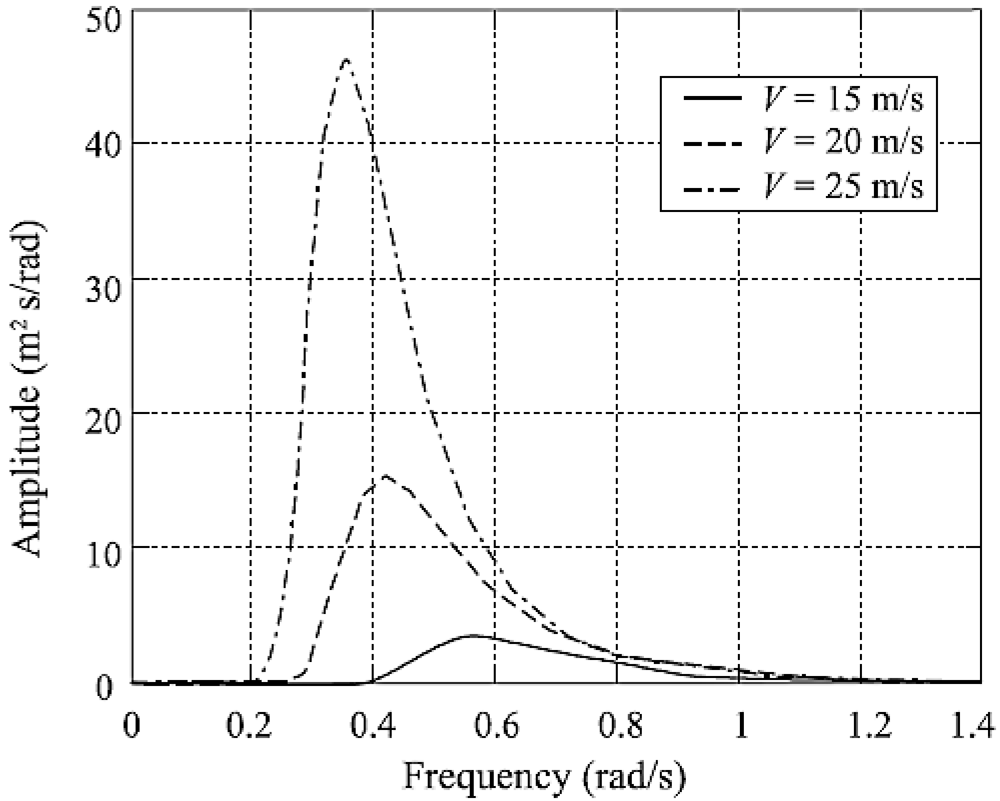
$$R_{XX}(\tau) = \sum_{n=1}^{N} \sigma_n^2 \cos \frac{2\pi n}{T}\tau,$$

Pierson Moskowitz Spectrum

Ocean waves generated by wind are modeled as random processes. A much used spectral density of ocean wave elevation $\eta(t)$ is the Pierson-Moskowitz spectrum,

$$S_{\eta\eta}(\omega) = \frac{0.0081g^2}{\omega^5} \exp\left[-0.74\left(\frac{g}{V\omega}\right)^4\right] \quad \mathrm{m}^2\mathrm{s},$$

where $\omega > 0$, g is the gravitational constant, and V is the wind speed at a height of 19.5 m above the still water level. Any consistent set of units for g and V can be used, with ω in rad/s.



50
40
30
20
10
0
Amplitude (m² s/rad)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
Frequency (rad/s)
V = 15 m/s
V = 20 m/s
V = 25 m/s

Example

Consider a process with the spectral density

$$S_{XX}(\omega) = \frac{1}{2(\omega_2 - \omega_1)} \text{ m}^2\text{s}, \text{ for } \omega_1 < |\omega| < \omega_2.$$

Find sample responses for (a) $\omega_1 = 10$ rad/s and $\omega_2 = 11$ rad/s,

- Step-1 $S_{XX}(\omega) = \frac{1}{2} \text{ m}^2\text{s}$
- Let $\Delta\omega = 0.1$ rad/s. $T = \frac{2\pi}{\Delta\omega} = \frac{2\pi}{0.1} = 62.83$ s.
- $\sqrt{2}\sigma_n = \sqrt{2}\sqrt{\frac{1}{2}0.1}$

Example

Choose 10 random uniform variates on $[0,2\pi)$ for φ_n. Let $N = 10$. From the random number table, using $\varphi_i = 0 + p_i(2\pi - 0)$,

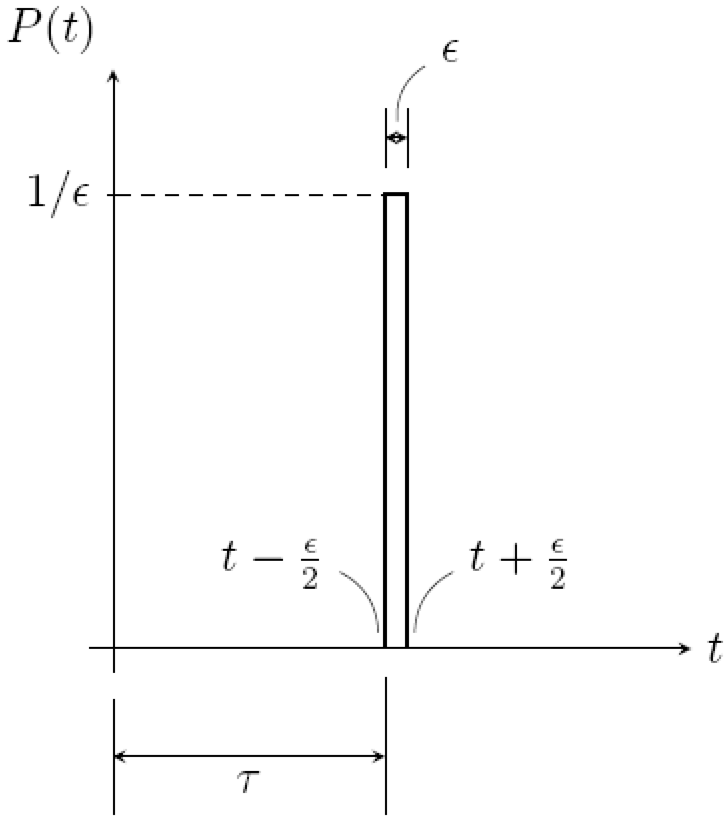
$$\begin{aligned}
\varphi_1 &= 0.34363(2\pi) = 2.159 \text{ rad} \\
\varphi_2 &= 0.79718(2\pi) = 5.009 \text{ rad} \\
&\vdots \\
\varphi_{10} &= 0.66529(2\pi) = 4.180 \text{ rad.}
\end{aligned}$$

$$\begin{aligned}
X(t) &= \sqrt{2}\sqrt{\frac{1}{2}0.1}\sum_{n=1}^{N}\cos\left(\omega_n t - \varphi_n\right) \\
&= 0.3162[\cos\left(10.05t - 2.159\right) + \cos\left(10.15t - 5.009\right) + \cdots \\
&\quad + \cos\left(10.95t - 4.180\right)]
\end{aligned}$$

Matlab code

```
clc; clear all; close all

 del_omega=0.1;   % delta-omega = 0.1;

omega1= 10.05: del_omega : 10.95;
 T = 2*pi/del_omega;   % Note that T = 2 * pi / delta-omega
 del_t= T/100;

t1=   0:del_t:T;

Nens=1;   % No of realizations

phi1=2*pi*rand(Nens, length(omega1));
 X1=zeros(length(t1),Nens);

for jj=1:Nens
    for i=1:length(t1)
        for j=1:length(omega1)
            [omega1' phi1']
            X1(i,jj)=X1(i,jj)+0.3162*cos(omega1(1,j)*t1(1,i)-phi1(jj,j));
        end
    end
end
figure, subplot(2,1,1), plot(t1,X1), xlabel('t (secs)'), ylabel('X(t) (m)')
title('Sample Time History for Sxx(w)=0.5 m^2’)
```

SDOF system under Impulse



Dirac delta function

$$\delta(t-\tau)=\begin{cases}+\infty, & ;t=\tau\\ 0 & ;t\neq\tau\end{cases}$$

$$\int_{-\infty}^{\infty}\delta(t-\tau)\ dt=1\ \ =m\Delta\dot{x}(t=\tau)$$

$$\dot{x}(\tau)=\frac{1}{m}$$

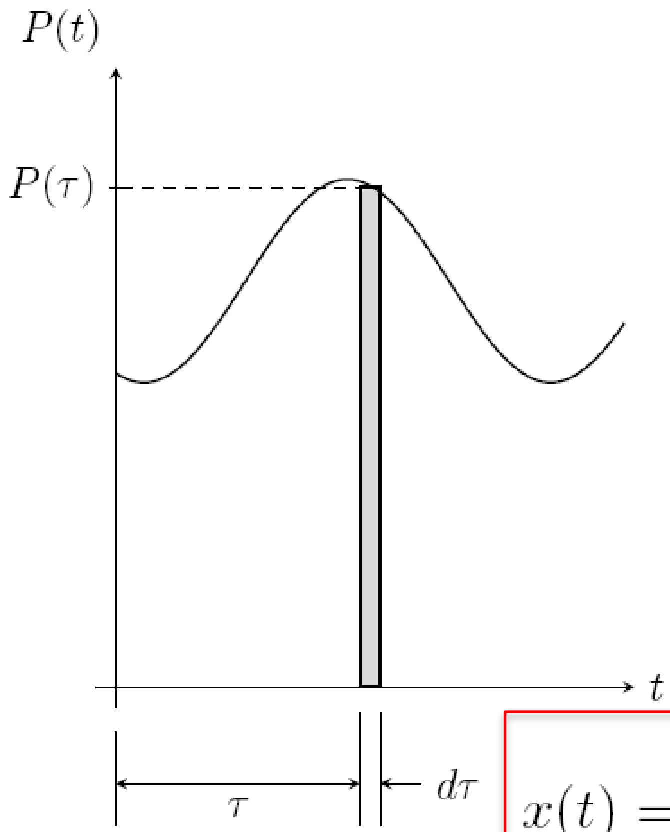
SDOF Impulse response

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = \delta(t-\tau)$$

Solution:

$$\boxed{x(t-\tau) = \frac{1}{m\omega_D} e^{-\zeta\omega_n(t-\tau)} \sin \omega_D (t-\tau)}$$

Convolution integral



$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = P(t)$$

$$dx = \frac{P(\tau)d\tau}{m\omega_D} e^{-\zeta\omega_n(t-\tau)} \sin\omega_D (t-\tau)$$

$$\boxed{x(t) = \frac{1}{m\omega_D} \int_0^t P(\tau) e^{-\zeta\omega_n(t-\tau)} \sin\omega_D (t-\tau) \; d\tau}$$

Frequency Response

Approach the solution to the equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t).$$

using Fourier transforms

$$\left[(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2\right] X(\omega) = F(\omega)$$

Therefore,

$$X(\omega) = \frac{F(\omega)}{(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2}$$

Frequency Response

- The quantity $\boxed{\dfrac{X(\omega)}{F(\omega)} = H(i\omega) = \dfrac{1}{(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2}}$

  is called frequency response function

Now consider the same equation of motion with

$$F(t) = \delta(t)$$

Impulse Response

- $F(t) = \delta(t)$
- By def the response to a unit impulse load is called the impulse response function and is denoted as $g(t)$
  The equation of motion is given by
- $$\ddot{g} + 2\zeta\omega_n\dot{g} + \omega_n^2 g = \delta(t).$$

  Take the Fourier transform of the equation of motion,

$$G(\omega) = \frac{1}{2\pi\left[(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2\right]}$$

Impulse Response

- $$G(\omega) = \frac{1}{2\pi\left[(i\omega)^2 + i2\zeta\omega_n\omega + \omega_n^2\right]}$$
- Then

$$X(\omega) = 2\pi G(\omega) F(\omega)$$

and $G(\omega)$ is related to $H(\omega)$ by

- $$H(\omega) = 2\pi G(\omega)$$

$$\boxed{H(\omega) = \int_{-\infty}^{\infty} g(t) \exp(-i\omega t)\, dt}$$

LINEAR SYSTEM RESPONSE TO STOCHASTIC EXCITATION

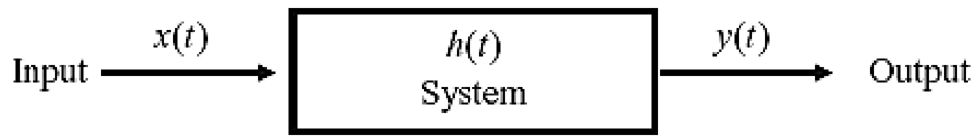


Figure A single-input, single-output system

$$y(t) = \int_{-\infty}^{t} h(t - t_1)x(t_1)dt_1$$

$$\mu_y = \mu_x \int_{0}^{\infty} h(\tau)d\tau$$

Linear system response to stochastic excitation: Autocorrelation

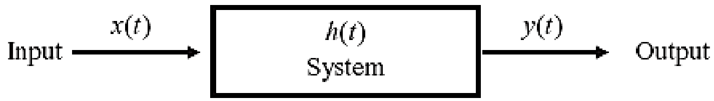


Figure A single-input, single-output system

$$y(t) = \int_{-\infty}^{t} h(t - t_1)x(t_1)dt_1$$

$$R_{yy}(\tau) = E[y(t)y(t+\tau)] = E\left[\int_0^{\infty}\int_0^{\infty} h(\tau_1)x(t-\tau_1)h(\tau_2)x(t+\tau-\tau_2)d\tau_1 d\tau_2\right]$$

$$= \int_0^{\infty}\int_0^{\infty} h(\tau_1)h(\tau_2)E\left[x(t-\tau_1)x(t+\tau-\tau_2)\right]d\tau_1 d\tau_2$$

$$\boxed{R_{yy}(\tau) = \int_0^{\infty}\int_0^{\infty} h(\tau_1)h(\tau_2)R_{xx}(\tau+\tau_1-\tau_2)d\tau_1 d\tau_2}$$

LINEAR SYSTEM RESPONSE TO STOCHASTIC EXCITATION: AUTO-SPECTRAL DENSITY

- Taking the Fourier transform of autocorrelation

$$S_{yy}(f) = \int_{-\infty}^{\infty} R_{yy}(\tau) e^{-j2\pi f \tau} d\tau$$

$$= \int_{0}^{\infty} h(\tau_1) e^{j2\pi f \tau_1} d\tau_1 \int_{0}^{\infty} h(\tau_2) e^{-j2\pi f \tau_2} d\tau_2 \int_{-\infty}^{\infty} R_{xx}(\tau + \tau_1 - \tau_2) e^{-j2\pi f (\tau + \tau_1 - \tau_2)} d\tau$$

Let $\tau + \tau_1 - \tau_2 = u$ in the last integral to yield

$$S_{yy}(f) = |H(f)|^2 S_{xx}(f)$$

LINEAR SYSTEM RESPONSE TO STOCHASTIC EXCITATION: CROSS CORRELATION &SPECTRUM

$$R_{xy}(\tau) = \int_{0}^{\infty} h(\tau_1) R_{xx}(\tau - \tau_1) d\tau_1 \qquad \textbf{(1)}$$

The Fourier transform of Equation **(1)** gives the frequency domain equivalent as

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j2\pi f \tau} d\tau = \int_{0}^{\infty} h(\tau_1) e^{-j2\pi f \tau_1} d\tau_1 \int_{-\infty}^{\infty} R_{xx}(\tau - \tau_1) e^{-j2\pi f (\tau - \tau_1)} d\tau$$

thus

$$S_{xy}(f) = H(f) S_{xx}(f)$$