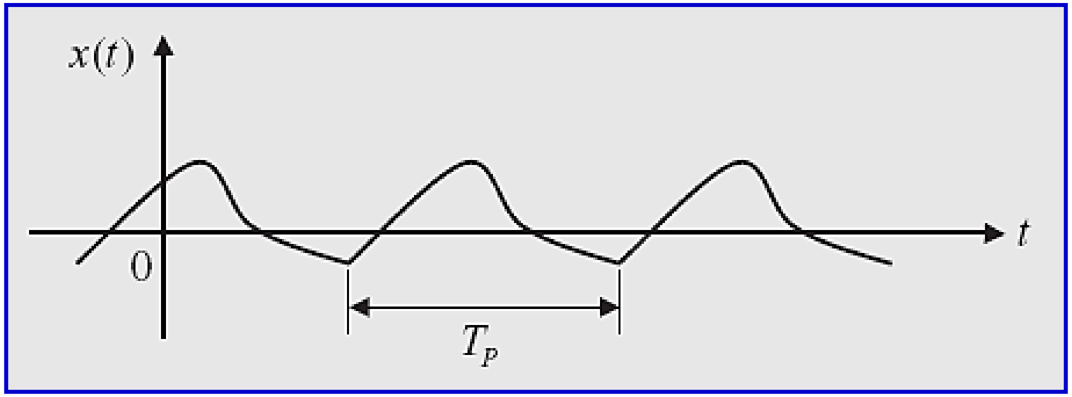
CE 513: STATISTICAL METHODS IN CIVIL ENGINEERING

Lecture: Introduction to Fourier transforms

Dr. Budhaditya Hazra
Room: N-307
Department of Civil Engineering

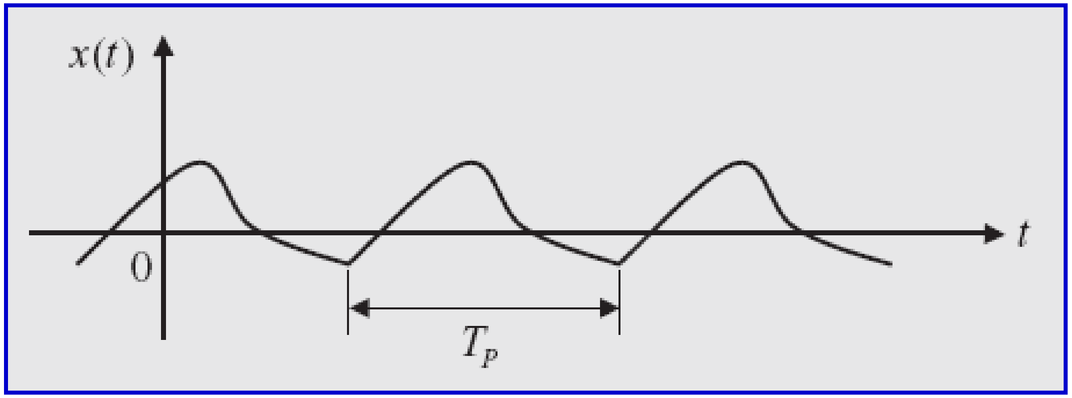
Fourier Analysis



A period signal with a period T_P

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi nt}{T_P}\right) + b_n \sin\left(\frac{2\pi nt}{T_P}\right)\right]$$

Fourier Series



$$\frac{a_0}{2} = \frac{1}{T_P}\int_0^{T_P} x(t)dt = \frac{1}{T_P}\int_{-T_P/2}^{T_P/2} x(t)dt \quad : \quad \text{mean value}$$

$$a_n = \frac{2}{T_P}\int_0^{T_P} x(t)\cos\left(\frac{2\pi nt}{T_P}\right)dt = \frac{2}{T_P}\int_{-T_P/2}^{T_P/2} x(t)\cos\left(\frac{2\pi nt}{T_P}\right)dt \quad n = 1,\ 2,\ \ldots$$

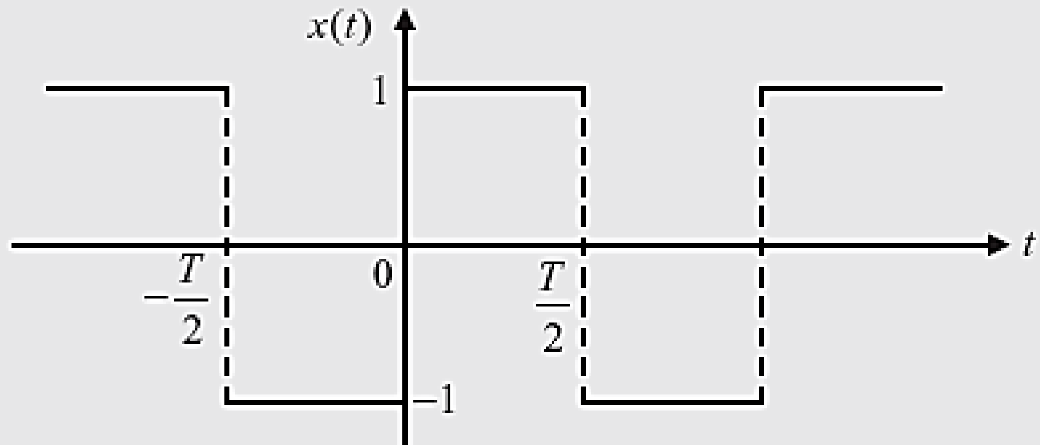
$$b_n = \frac{2}{T_P}\int_0^{T_P} x(t)\sin\left(\frac{2\pi nt}{T_P}\right)dt = \frac{2}{T_P}\int_{-T_P/2}^{T_P/2} x(t)\sin\left(\frac{2\pi nt}{T_P}\right)dt \quad n = 1,\ 2,\ \ldots$$

Fourier Series: Example

$$x(t) = -1 \quad -\frac{T}{2} < t < 0$$

$$\quad\quad = 1 \quad 0 < t < \frac{T}{2}$$

and $x(t + nT) = x(t) \quad n = \pm 1, \pm 2, \ldots$



Fourier Series: Example

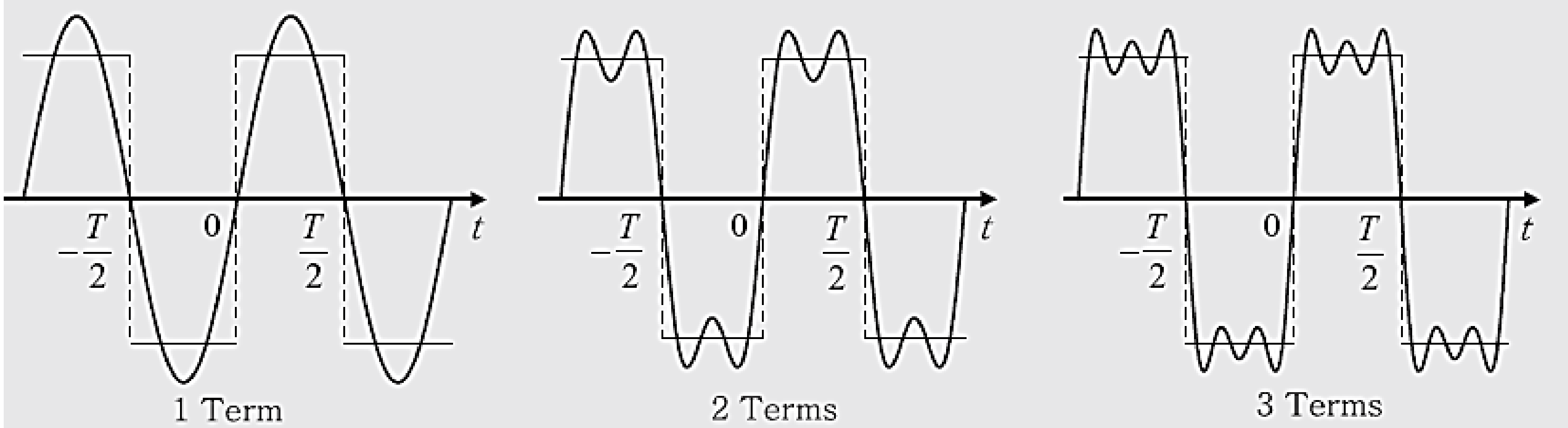
$$a_0 = 0; \quad a_n = 0$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin\left(\frac{2n\pi t}{T}\right) dt = \frac{2}{n\pi} (1 - n\pi)$$

$$x(t) = \frac{4}{\pi}\left[\sin\left(\frac{2\pi t}{T}\right) + \frac{1}{3}\sin\left(\frac{2\pi 3t}{T}\right) + \frac{1}{5}\sin\left(\frac{2\pi 5t}{T}\right) + \cdots\right]$$

$$x(t) = \frac{4}{\pi}\left[\sin \omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t + \cdots\right]$$

Fourier Series: Example



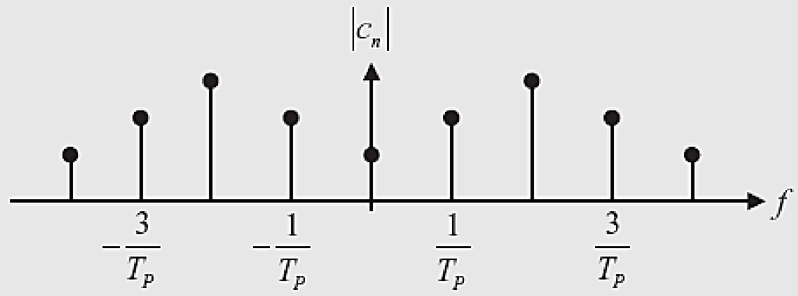
$$S_1(t) = \frac{4}{\pi} \sin \omega_1 t$$

$$S_2(t) = \frac{4}{\pi}\left[\sin \omega_1 t + \frac{1}{3}\sin 3\omega_1 t\right]$$

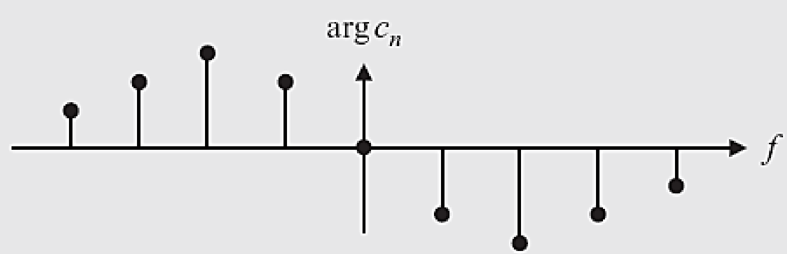
$$S_3(t) = \frac{4}{\pi}\left[\sin \omega_1 t + \frac{1}{3}\sin 3\omega_1 t + \frac{1}{5}\sin 5\omega_1 t\right]$$

Amplitude & Phase Spectrum

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{\frac{j2\pi nt}{T}} \qquad\qquad c_n = \frac{2}{T}\int_0^T x(t)\; e^{-\frac{j2\pi nt}{T}} \; dt$$



Amplitude spectrum of a Fourier series (a *line spectrum* and an *even function*)



Phase spectrum of a Fourier series (a *line spectrum* and an *odd function*)

MATLAB: example

```
clc; close all; clear all

 % Keep hitting enter button if you want to see the term by term approx.
 t=[0:0.001:1];
 x=[ ]; x_tmp=zeros(size(t));

for n=1:2:39
        x_tmp=x_tmp+4/pi*(1/n*sin(2*pi*n*t));
        x=[x; x_tmp];
end

figure,
 for i=1:20
     drawnow
        plot(t, x(i,:))   %plot(t,x(i,:),t,x(7,:),t,x(20,:));
        xlabel('\itt\rm (seconds)'); ylabel('\itx\rm(\itt\rm)')
        grid on
    pause
 end
```

FOURIER TRANSFORM

- Extension of Fourier analysis to non-periodic phenomena
- Discrete to continuous
- Skipping essential steps, in the limit $T_p \rightarrow \infty$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

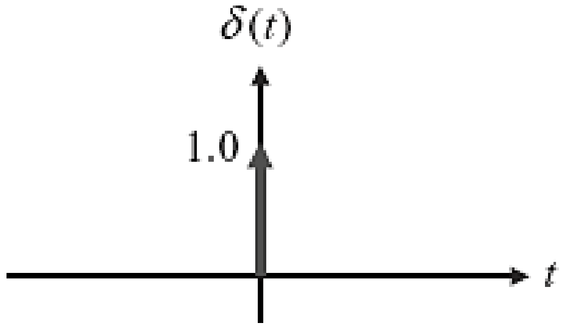
Fourier transform

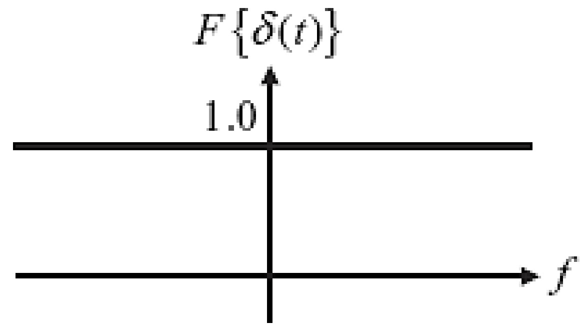
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

Inverse
Fourier transform

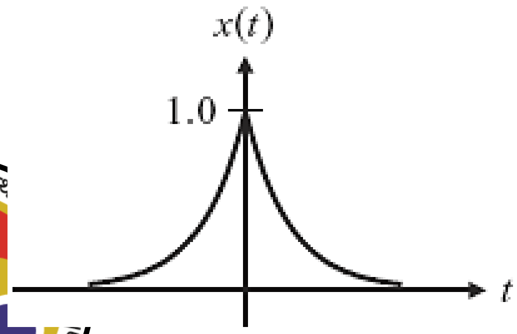
Examples

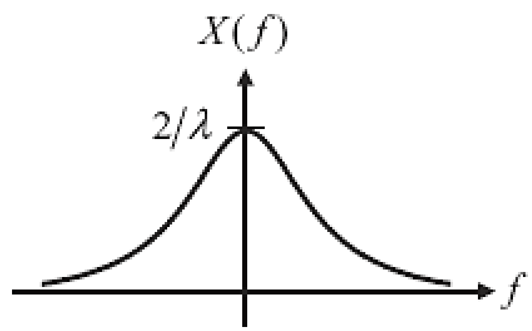
1) Dirac delta





2) Symmetric exponential $x(t) = e^{-\lambda|t|}, \quad \lambda > 0$



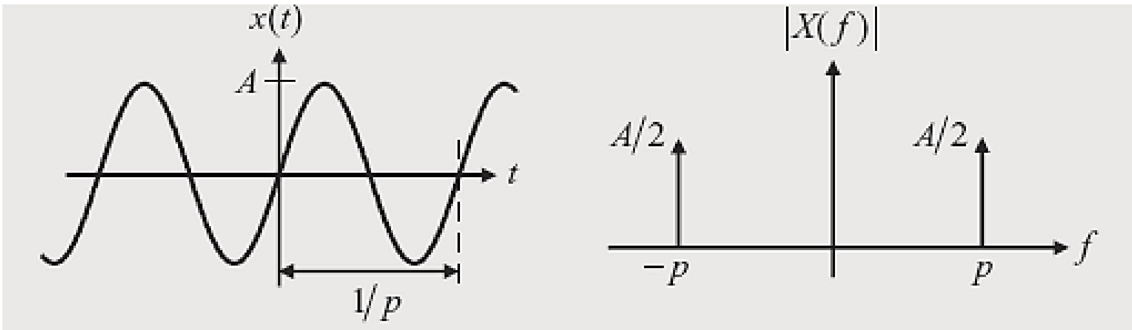


$$X(f) = \frac{2\lambda}{\lambda^2 + 4\pi^2 f^2}$$

Examples

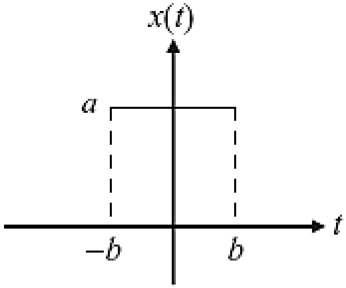
3) Sinusoid

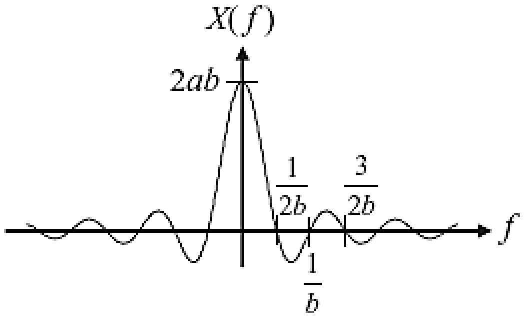
$$x(t) = \sin(2\pi f_0 t) \text{ or } \sin(\omega_0 t) \quad X(f) = \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$



4) Window function

$$x(t) = a \quad |t| < b$$
$$= 0 \quad |t| > b$$





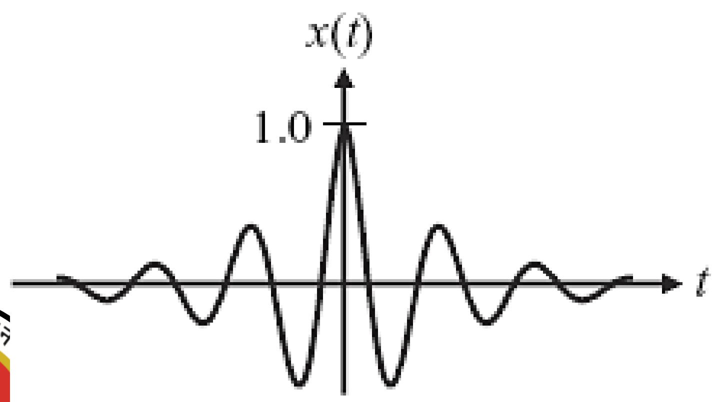
$$X(f) = \frac{2ab\sin(2\pi f b)}{2\pi f b}$$

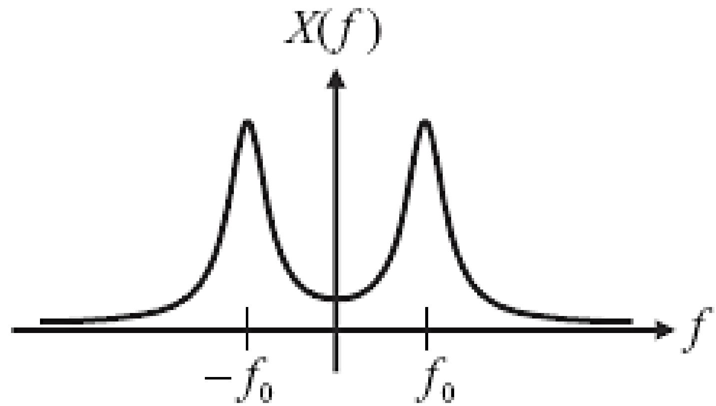
Examples

5) Damped symmetrically oscillating function

$$x(t) = e^{-a|t|} \cos 2\pi f_0 t, \quad a > 0$$

$$X(f) = \frac{a}{a^2 + [2\pi(f - f_0)]^2} + \frac{a}{a^2 + [2\pi(f + f_0)]^2}$$

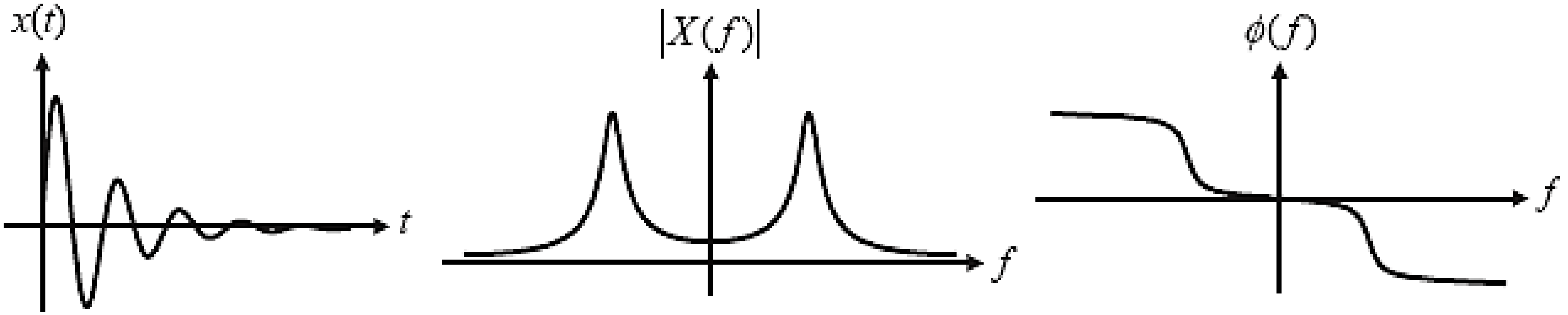




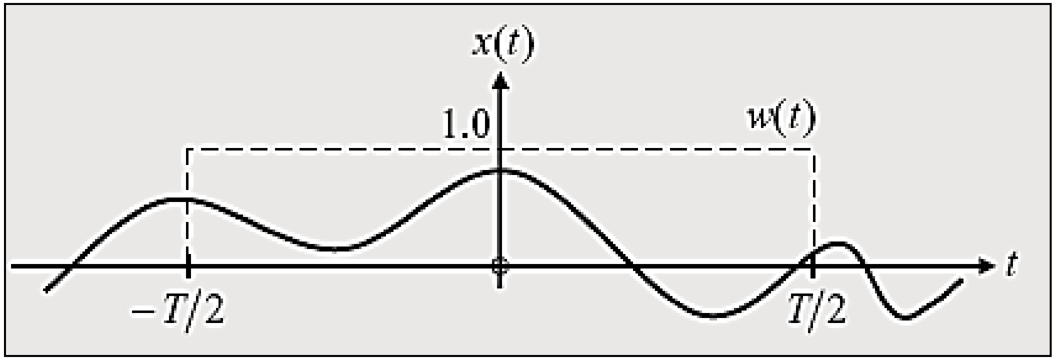
6) Damped oscillating function

$$x(t) = e^{-at} \sin 2\pi f_0 t, \quad t \geq 0 \text{ and } a > 0$$

$$X(f) = \frac{2\pi f_0}{(2\pi f_0)^2 + (a + j2\pi f)^2}$$



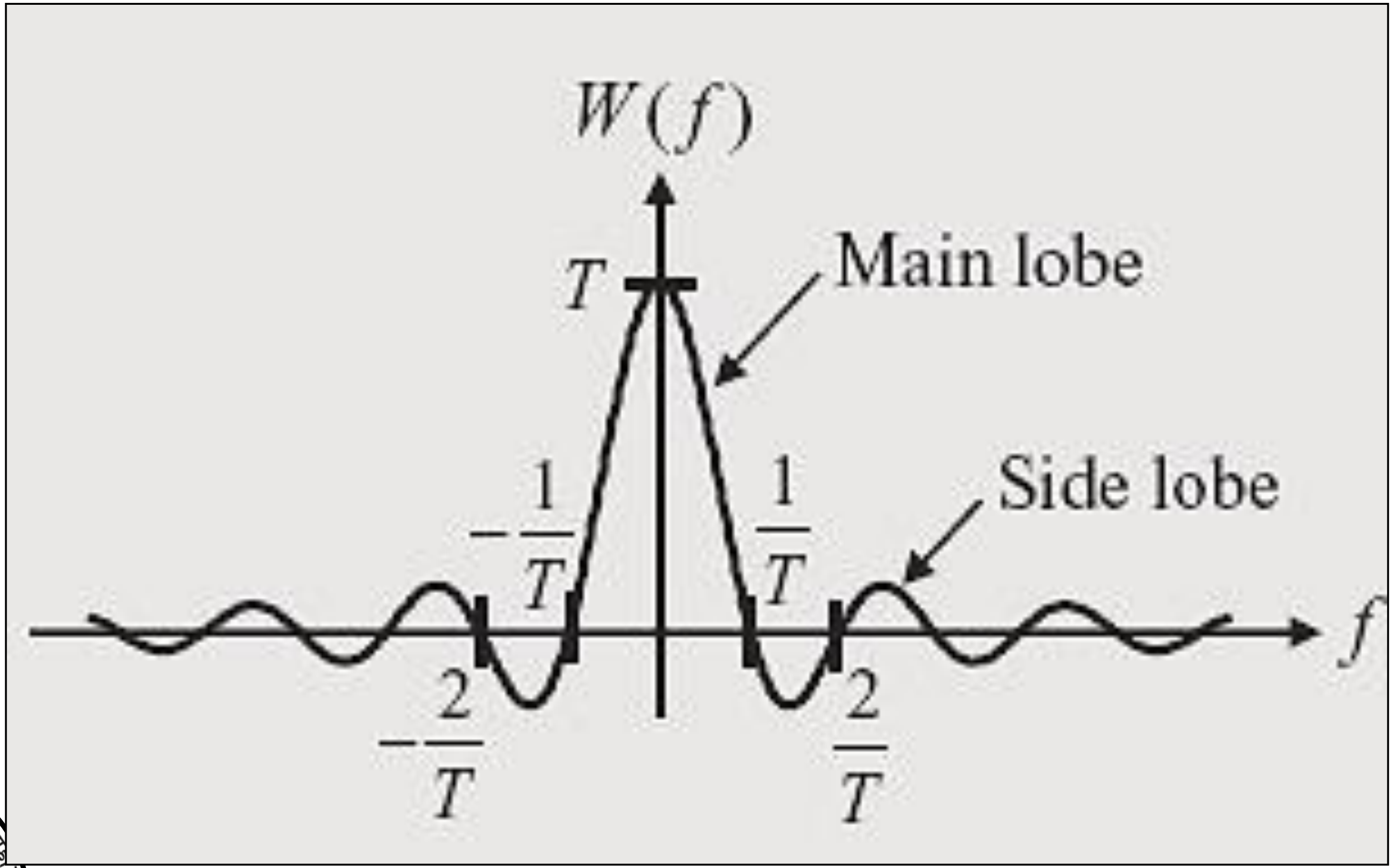
Windowing



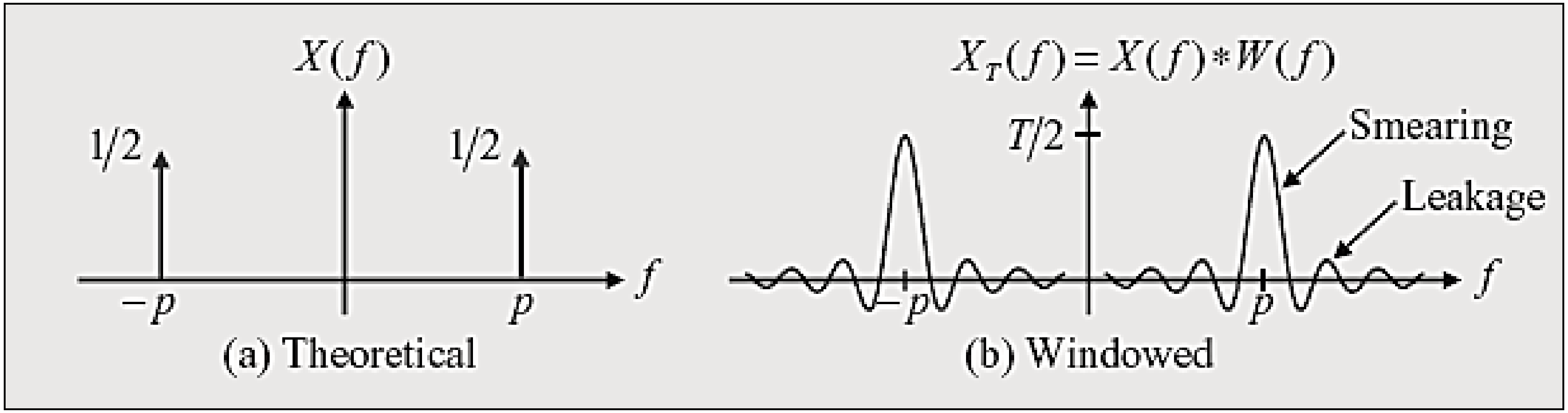
$$w(t) = 1 \quad |t| < T/2$$
$$= 0 \quad |t| > T/2$$

$$X_T(f) = \int_{-\infty}^{\infty} X(g)W(f-g)dg = \frac{1}{2}\int_{-\infty}^{\infty} [\delta(g+p)+\delta(g-p)]\,W(f-g)dg$$
$$= \frac{1}{2}[W(f+p)+W(f-p)]$$

Windowing



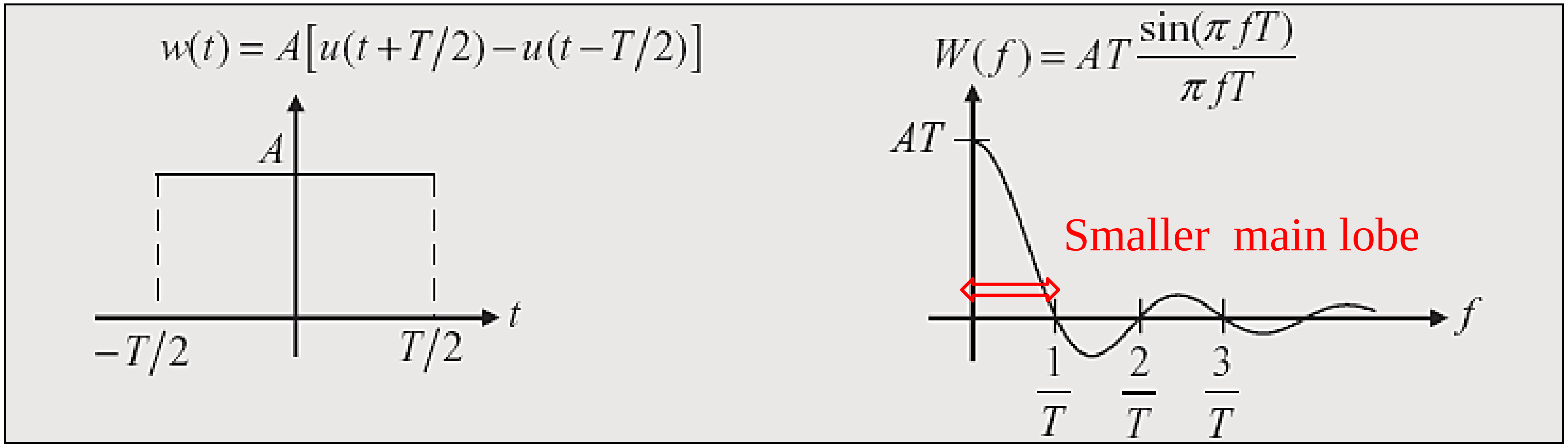
Windowing: Illustration



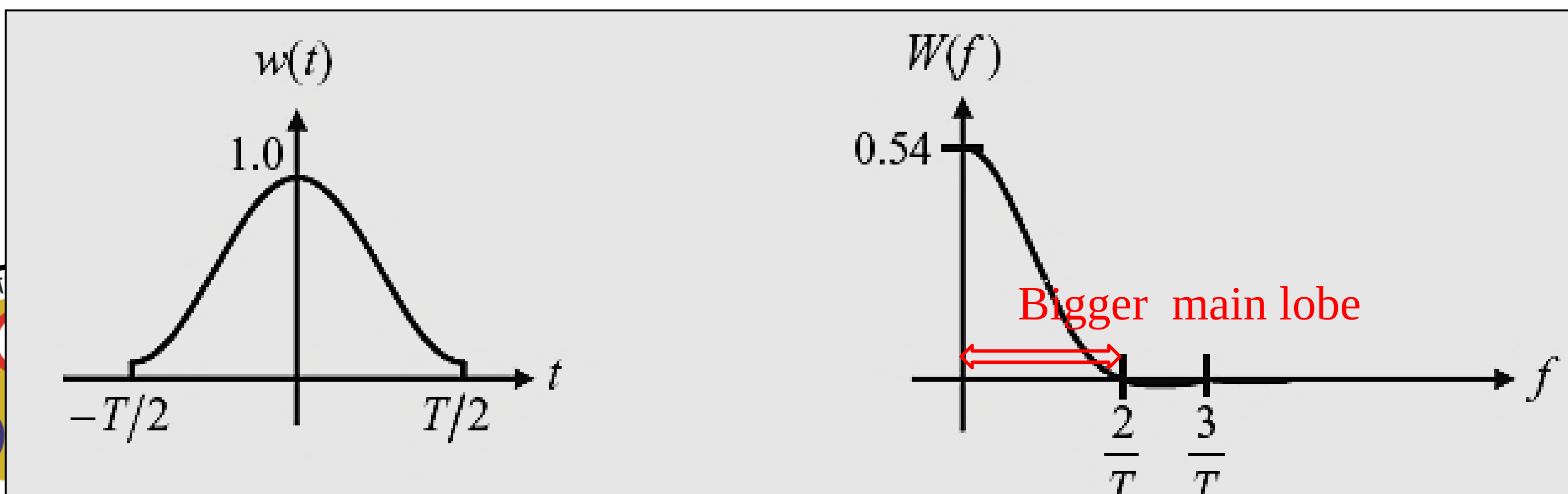
The distortion due to the main lobe is sometimes called *smearing*, and the distortion caused by the side lobes is called *leakage*.

Common window functions

Rectangular Window

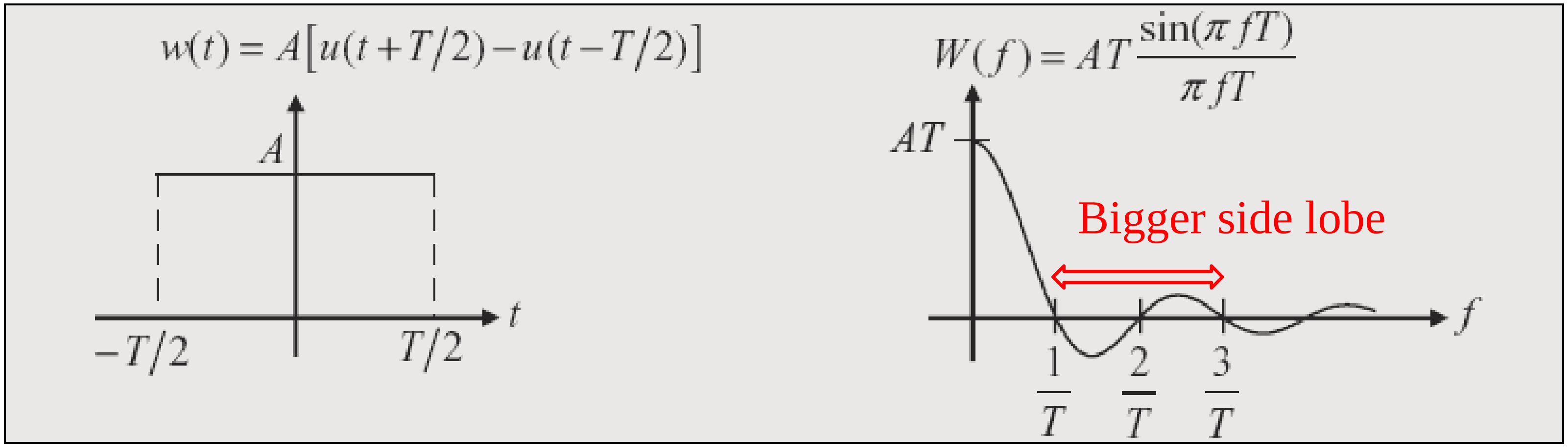


Hann Window

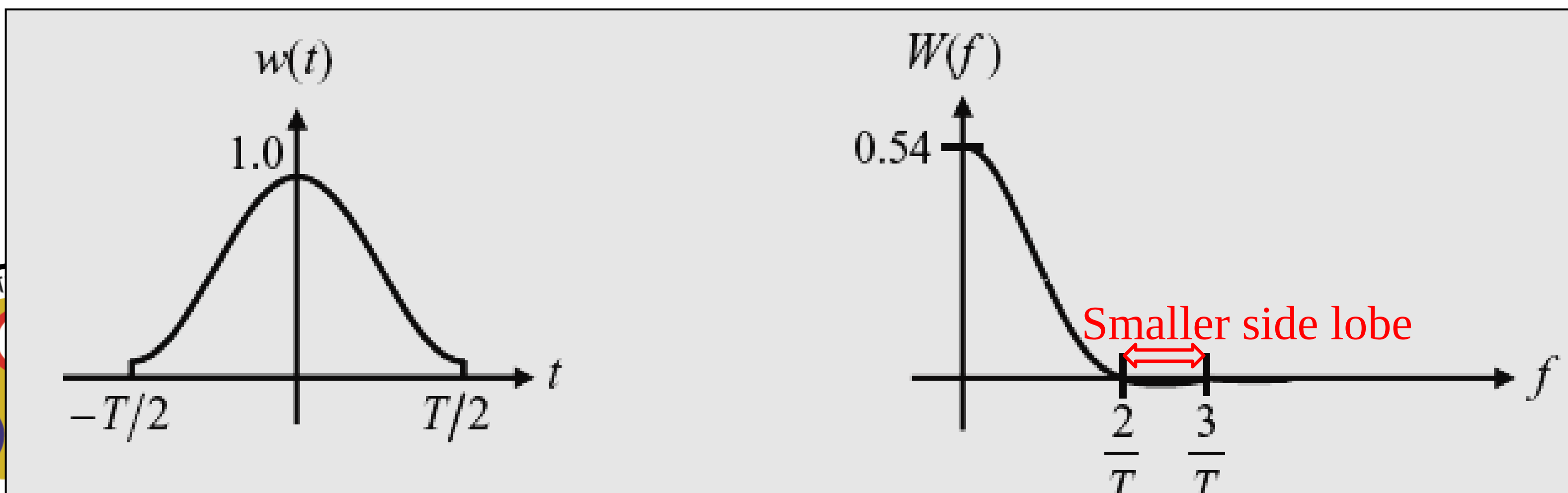


Common window functions

Rectangular Window



Hann Window



Discrete Fourier Transform

- Consider a sequence $x(n\Delta)$ at $n = 0, 1, 2, 3, 4, .., N-1$ points. The DFT is defined as :

$$X(e^{j2\pi f\Delta}) = \sum_{n=0}^{N-1} x(n\Delta)e^{-j2\pi f n\Delta}$$

- Note that this is still continuous in frequency

Discrete Fourier Transform

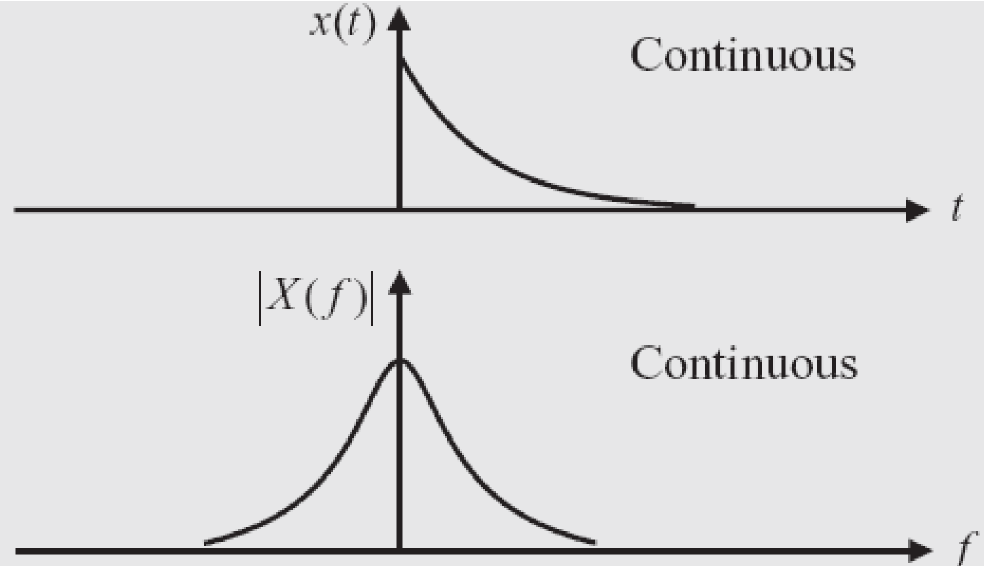
Now let us evaluate this at frequencies: $f = k / N\Delta$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk}$$

$$X(k) = \left[X(e^{j2\pi f\Delta}) \text{ evaluated at } f = \frac{k}{N\Delta} \text{Hz} \right] (k \text{ integer})$$

Fourier Integral vs DFT

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

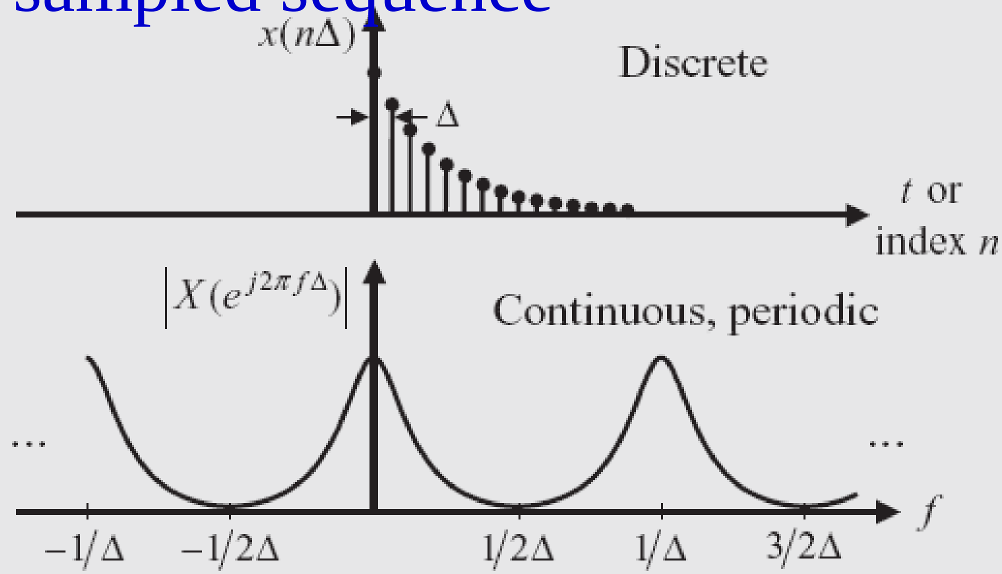


Fourier Integral

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Fourier transform of the sampled sequence

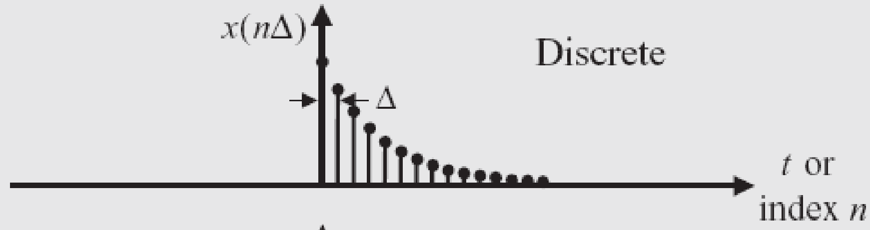
$$x(n\Delta) = \Delta \int_{-1/2\Delta}^{1/2\Delta} X(e^{j2\pi f\Delta}) e^{j2\pi fn\Delta} df$$



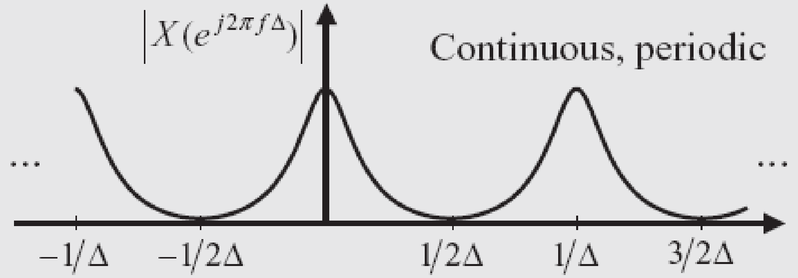
$$X(e^{j2\pi f\Delta}) = \sum_{n=-\infty}^{\infty} x(n\Delta) e^{-j2\pi fn\Delta}$$

Fourier Integral vs DFT

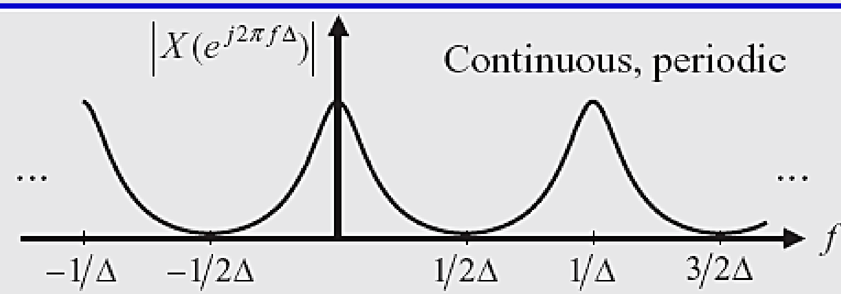
$$x(n\Delta) = \Delta \int_{-1/2\Delta}^{1/2\Delta} X(e^{j2\pi f\Delta}) e^{j2\pi fn\Delta} df$$



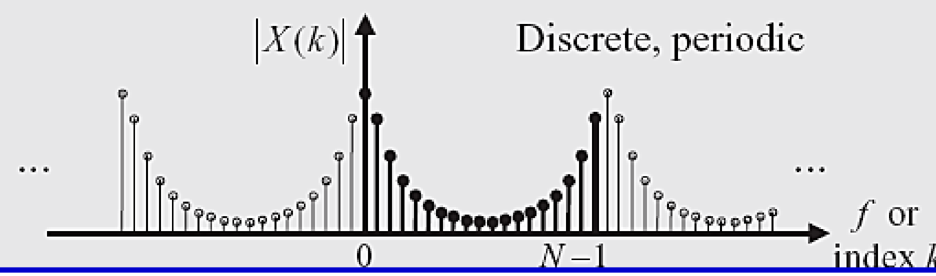
$$X(e^{j2\pi f\Delta}) = \sum_{n=-\infty}^{\infty} x(n\Delta) e^{-j2\pi fn\Delta}$$



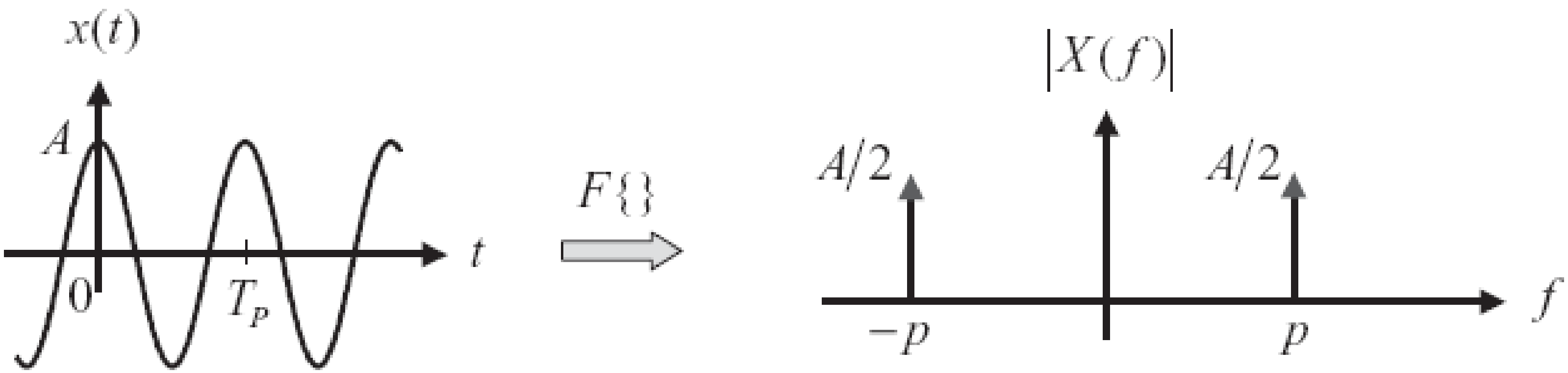
$$X(e^{j2\pi f\Delta}) = \sum_{n=-\infty}^{\infty} x(n\Delta) e^{-j2\pi fn\Delta}$$



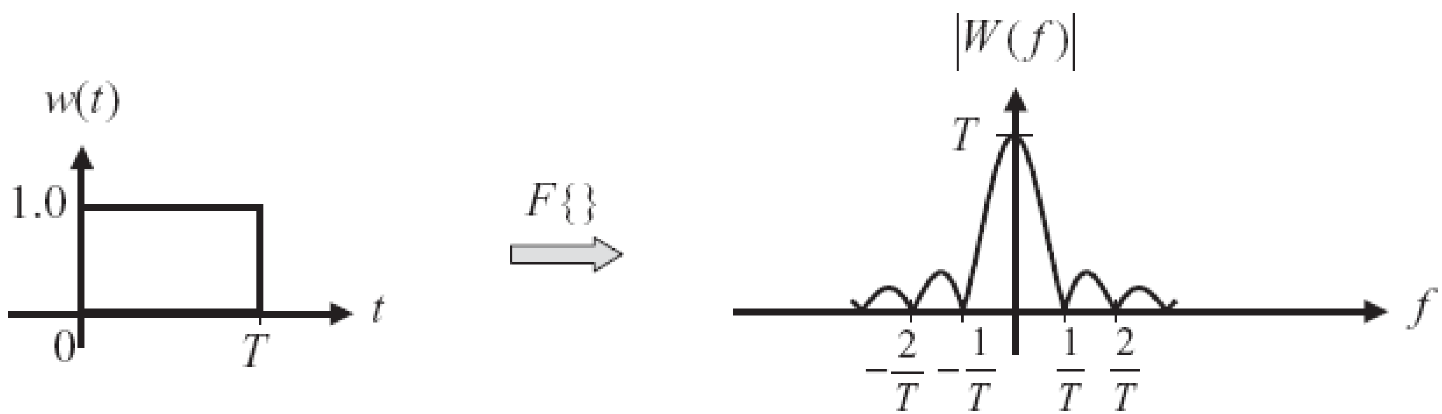
$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk}$$



Fourier Integral vs DFT



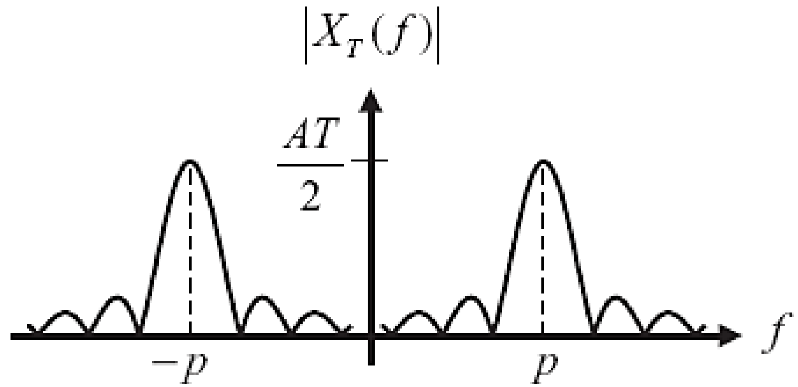
A periodic signal and its Fourier integral



Data window and its Fourier integral

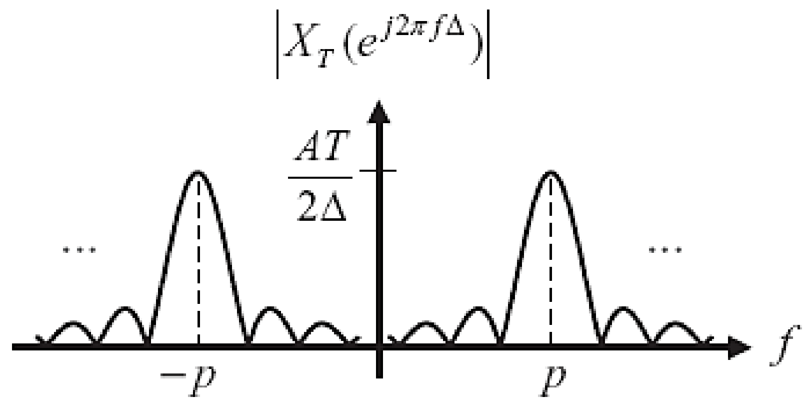
FOURIER INTEGRAL VS DFT

$$x_T(t) = w(t) \cdot x(t) \xrightarrow{F\{\}}$$



(c) Truncated signal and its Fourier integral

$$x_T(n\Delta) = w(n\Delta) \cdot x(n\Delta) \xrightarrow{F\{\}}$$



(d) Truncated and sampled signal and its Fourier transform of a sequence

FFT ALGORITHM: GLIMPSES

- The DFT provides uniformly spaced samples of the Discrete-Time Fourier Transform (DTFT)

- DFT definition:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}} \qquad x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] e^{j\frac{2\pi nk}{N}}$$

- Requires N^2 complex multiplications & N(N-1) complex additions

Let's take an example of FFT: a simple MATLAB code

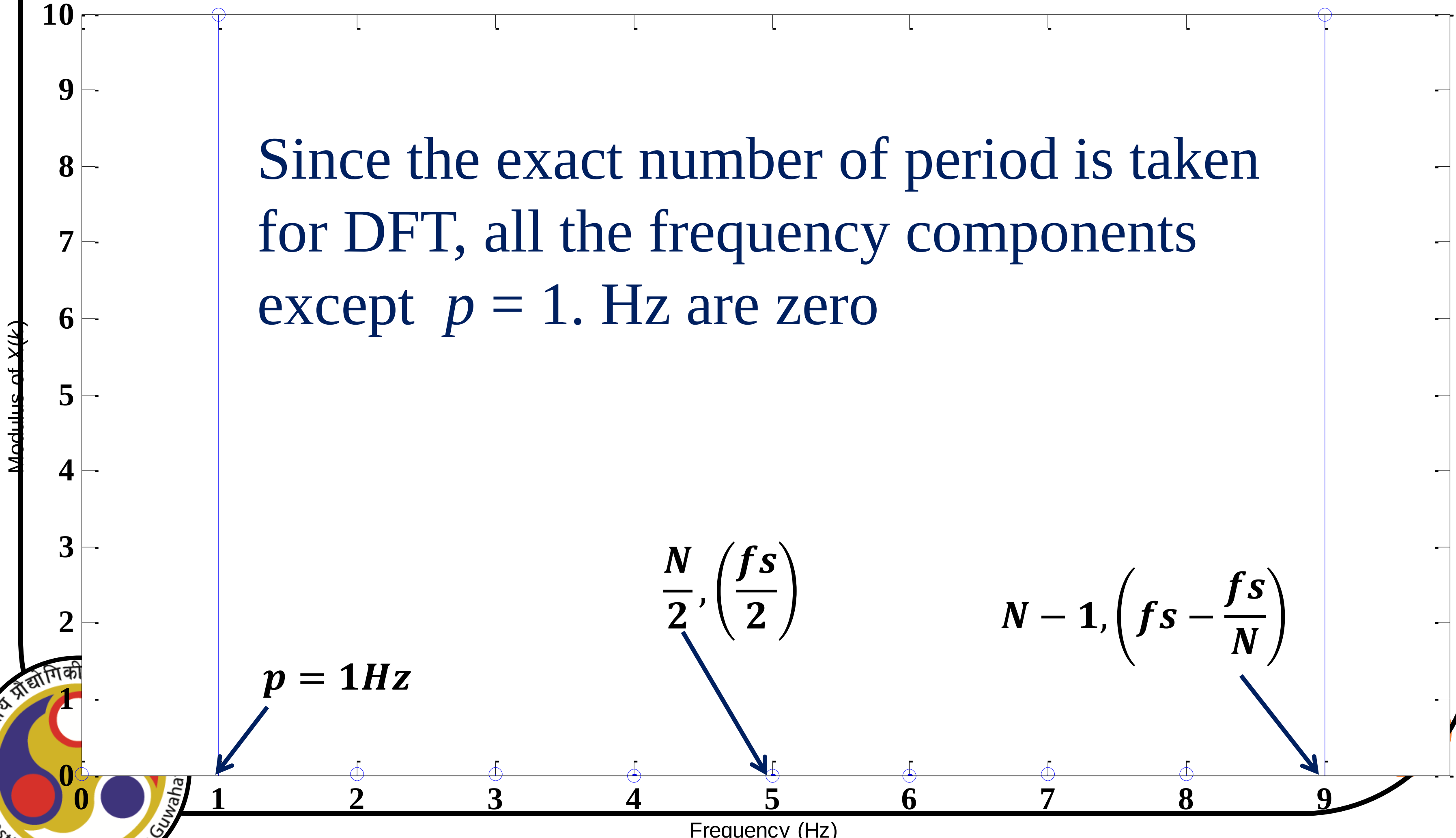
- Consider a signal: x = $A \sin 2\pi p t$
- p = $\frac{1}{T_p}$. Hz
- Set a sampling frequency fs = $\frac{10}{T_p}$. Hz
- A = 2, p =1 Hz
- What does Fourier integral give ? $|X(f)| = \frac{A}{2}$ at p = 1 Hz

FFT MATLAB EXAMPLE-1

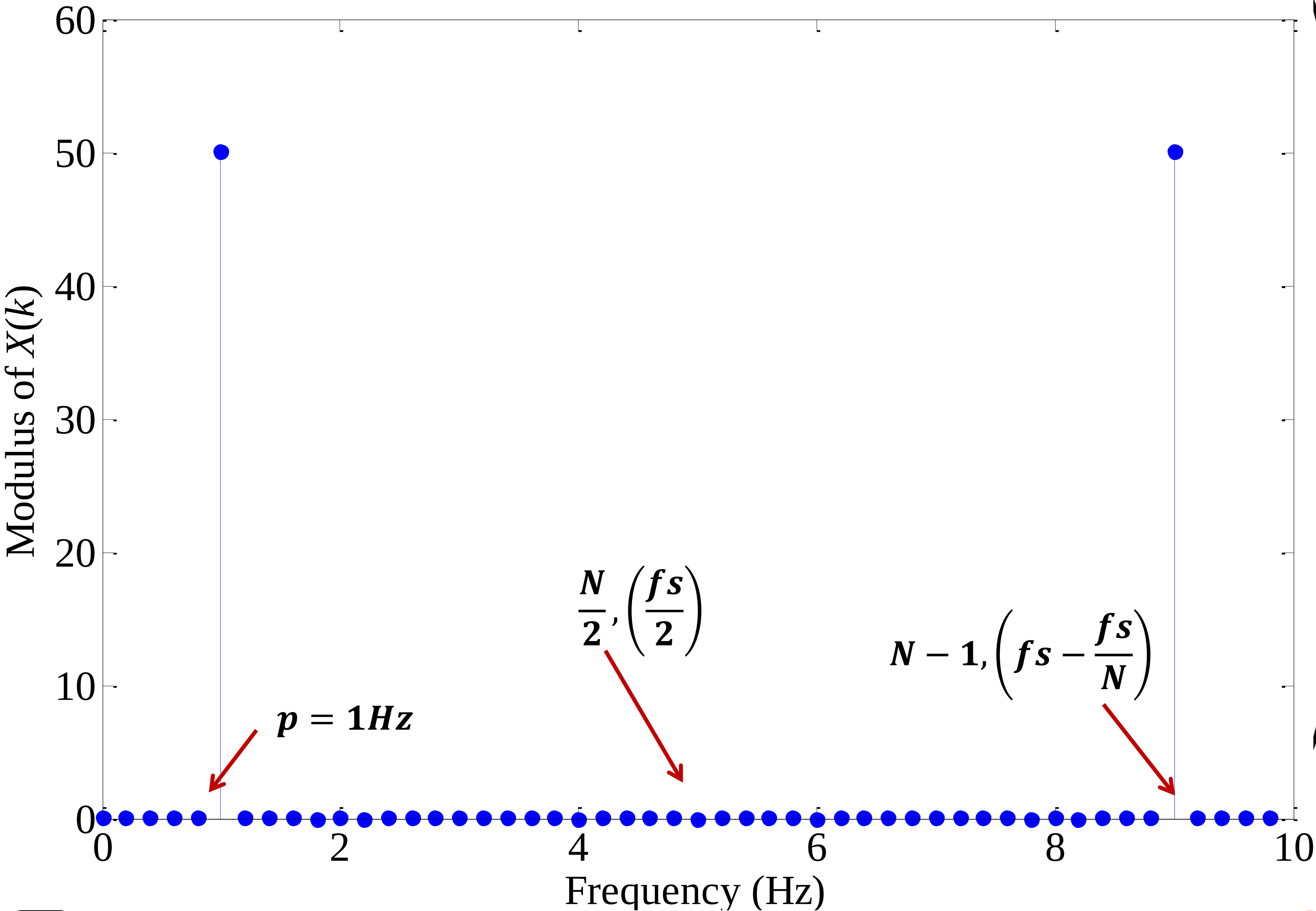
```
freq = 1;        % Frequency of sinusoid. freq= 1/ Tp;

Tp=1/freq;       % Tp is the period

fs=10*freq;      % Define sample time and sample frequency. Take fs = 10/Tp

ts=1/fs;

T1= 1*Tp;        % Time point where the data is truncated to

t=0:ts:T1-ts;

x= 2*cos(2*pi*freq*t);

 L=length(x);

FT = fft(x);   % FFT command; gives Fourier amplitude

f =fs* (0:(L-1))/L;  % Creating the X-axis or frequency axis

fabs=abs(FT);     % Single sided Fourier transform
```

DATA TRUNCATED AT EXACTLY ONE PERIOD (10-POINT FFT)

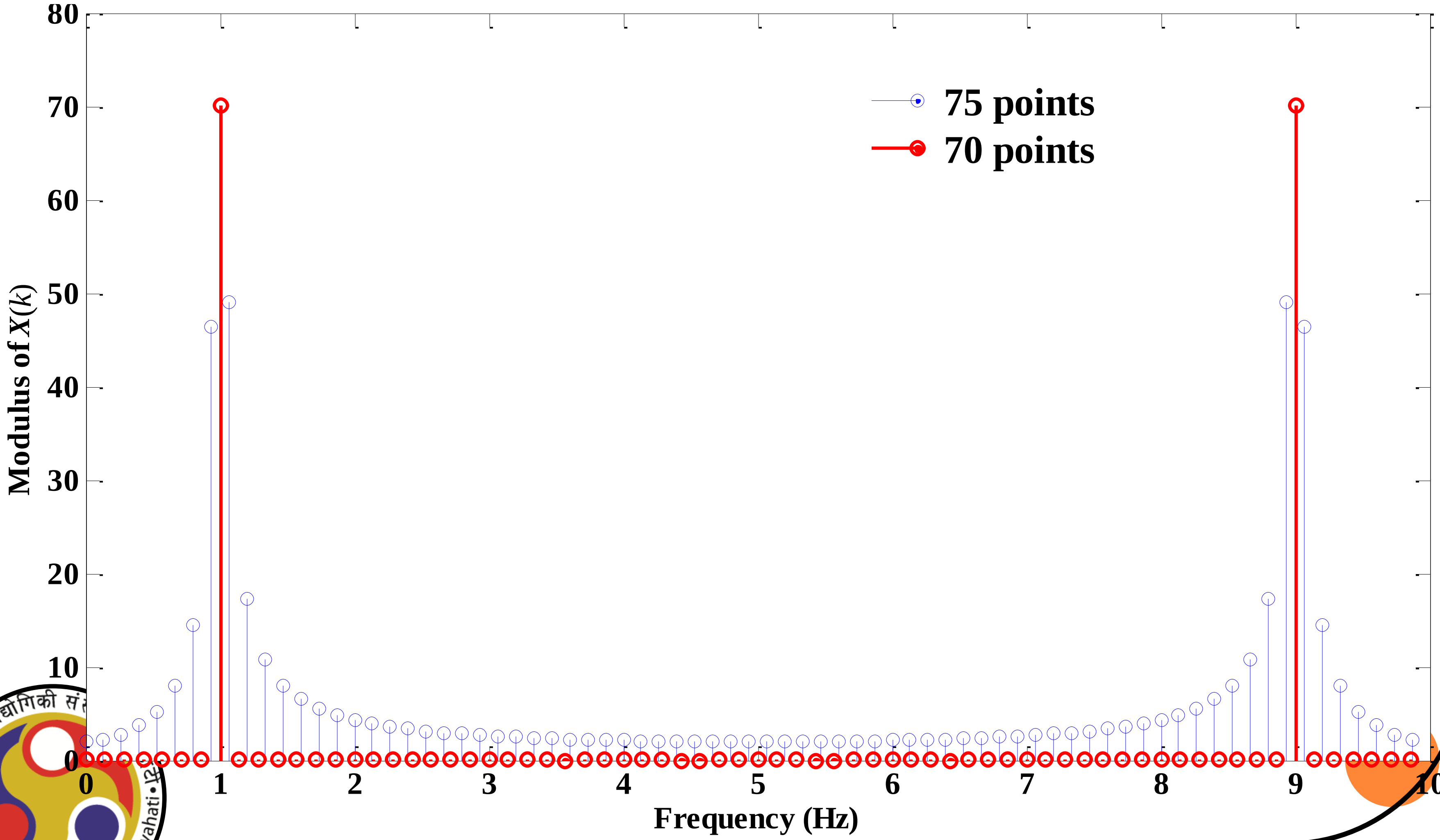
Since the exact number of period is taken for DFT, all the frequency components except $p = 1$. Hz are zero



DATA TRUNCATED AT 5 PERIODS (50-POINT FFT)



LEAKAGE


75 points
70 points
Modulus of X(k)
Frequency (Hz)
80
70
60
50
40
30
20
10
0
0
1
2
3
4
5
6
7
8
9
10

FFT MATLAB EXAMPLE-2

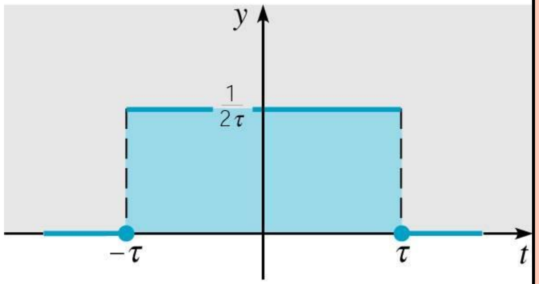
```
clc; clear all; close all
 A=2;
p=1;
Tp=1/p;
fs=10/Tp;

No_of_periods = 2;  %Keep changing this parameter

T1=No_of_periods*Tp;

 t1=[0:1/fs:T1-1/fs];

 x1=A*cos(2*pi*p*t1);

X1=fft(x1);

 N1=length(x1); f1=fs*(0:N1-1)/N1;

figure(1)
subplot(1,2,1)
plot(f1, abs(X1), 'b')
xlabel('Frequency (Hz)')
ylabel('Modulus of \itX\rm(\itk\rm)'); %axis([0 9.9 0 10])
 subplot(1,2,2)
stem(f1, abs(X1), 'b')
xlabel('Frequency (Hz)')
ylabel('Modulus of \itX\rm(\itk\rm)'); %axis([0 9.9 0 10]
```

EXTRA

Unit Impulse Function

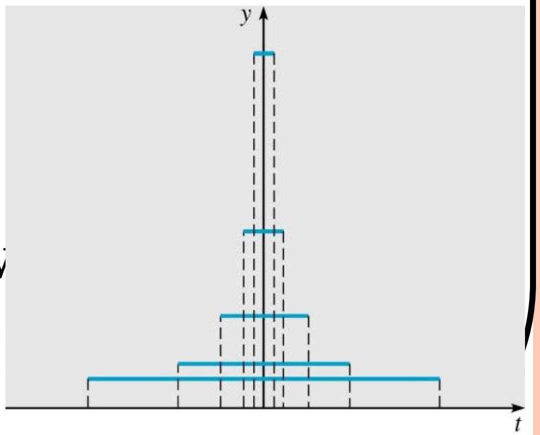
- Suppose a function $d_\tau(t)$ has the form

$$d_\tau(t) = \begin{cases} 1/2\tau, & -\tau < t < \tau \\ 0, & \text{otherwise} \end{cases}$$



- Then, $I(\tau) = 1$.

- We are interested $d_\tau(t)$ acting over shorter and shorter time intervals (i.e., $\tau \to 0$). See graph on right.



- Note that $d_\tau(t)$ gets taller and narrow as $\tau \to 0$. Thus for $t \neq 0$, we have $\lim_{\tau \to 0} d_\tau(t) = 0$, and $\lim_{\tau \to 0} I(\tau) = 1$

Dirac Delta Function

- Thus for $t \neq 0$, we have $\lim_{\tau \to 0} d_\tau(t) = 0$, and $\lim_{\tau \to 0} I(\tau) = 1$
- The **unit impulse function** δ is defined to have the properties

$$\delta(t) = 0 \text{ for } t \neq 0, \text{ and } \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

- The unit impulse function is an example of a generalized function and is usually called the **Dirac delta function**.
- In general, for a unit impulse at an arbitrary point t_0,

$$\delta(t - t_0) = 0 \text{ for } t \neq t_0, \text{ and } \int_{-\infty}^{\infty} \delta(t - t_0)\,dt = 1$$

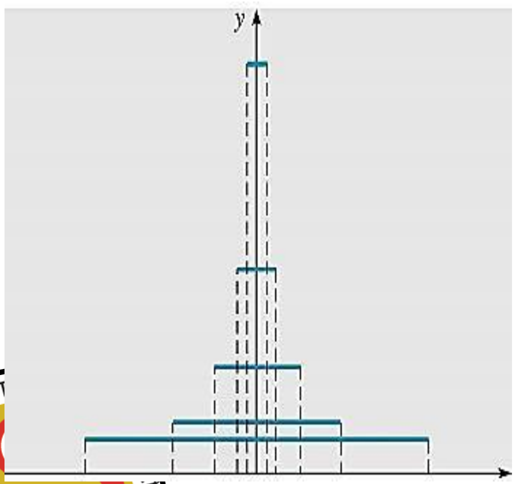
Laplace Transform of δ

- The Laplace Transform of δ is defined by

$$L\{\delta(t-t_0)\} = \lim_{\tau\to 0} L\{d_\tau(t-t_0)\},\ t_0 > 0$$

and thus

$$L\{\delta(t-t_0)\} = \lim_{\tau\to 0}\int_0^\infty e^{-st} d_\tau(t-t_0)dt = \lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}dt$$



$$= \lim_{\tau\to 0}\frac{-e^{-st}}{2s\tau}\Bigg|_{t_0-\tau}^{t_0+\tau} = \lim_{\tau\to 0}\frac{1}{2s\tau}\left[-e^{-s(t_0+\tau)} + e^{-s(t_0-\tau)}\right]$$

$$= \lim_{\tau\to 0}\frac{e^{-st_0}}{s\tau}\left[\frac{e^{s\tau}-e^{-s\tau}}{2}\right] = e^{-st_0}\left[\lim_{\tau\to 0}\frac{\sinh(s\tau)}{s\tau}\right]$$

$$= e^{-st_0}\left[\lim_{\tau\to 0}\frac{s\cosh(s\tau)}{s}\right] = e^{-st_0}$$

Laplace Transform of δ

- Thus the Laplace Transform of δ is

$$L\{\delta(t-t_0)\} = e^{-st_0}, \ t_0 > 0$$

- For Laplace Transform of δ at $t_0 = 0$, take limit as follows:

$$L\{\delta(t)\} = \lim_{t_0 \to 0} L\{d_\tau(t-t_0)\} = \lim_{\tau_0 \to 0} e^{-st_0} = 1$$

- For example, when $t_0 = 10$, we have $L\{\delta(t-10)\} = e^{-10s}$.