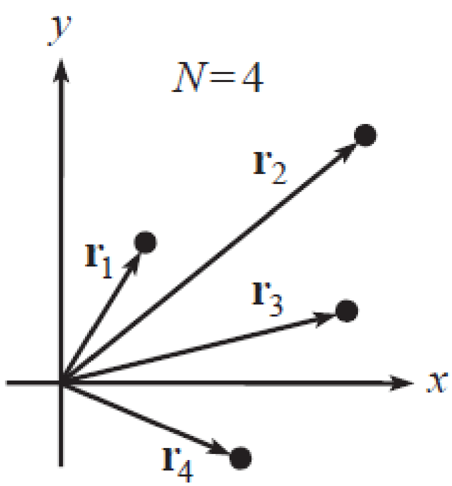
Lecture-XII

Angular momentum and Fixed axis rotation

Angular Momentum of a System of Particles

Consider a collection of N discrete particles. The total angular momentum of the system is

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i .$$



The force acting on each particle is $\mathbf{F}_i^{\rm ext} + \mathbf{F}_i^{\rm int} = d\mathbf{p}_i/dt.$

The internal forces come from the adjacent molecules which are usually central forces, so that the force between two molecules is directed along the line between them.

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}\sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i \frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \sum_i \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt}$$

$$= \sum_i \mathbf{v}_i \times (m\mathbf{v}_i) + \sum_i \mathbf{r}_i \times (\mathbf{F}_i^{\rm ext} + \mathbf{F}_i^{\rm int})$$

$$= 0 + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\rm ext} \equiv \sum_i \boldsymbol{\tau}_i^{\rm ext}.$$

$$\mathbf{F}_i^{\rm int} = \sum_i \mathbf{F}_{ij}^{\rm int}.$$

$$\boldsymbol{\tau}^{\rm int} \equiv \sum \mathbf{r}_i \times \mathbf{F}_i^{\rm int} = \sum\sum \mathbf{r}_i \times \mathbf{F}_{ij}^{\rm int}.$$

$$\boldsymbol{\tau}^{\rm int} = \sum_j\sum_i \mathbf{r}_j \times \mathbf{F}_{ji}^{\rm int} = -\sum_j\sum_i \mathbf{r}_j \times \mathbf{F}_{ij}^{\rm int},$$

$$\mathbf{F}_{ij}^{\rm int} = -\mathbf{F}_{ji}^{\rm int}$$

$$2\boldsymbol{\tau}^{\rm int} = \sum_i\sum_j (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij}^{\rm int} \cdot = 0$$

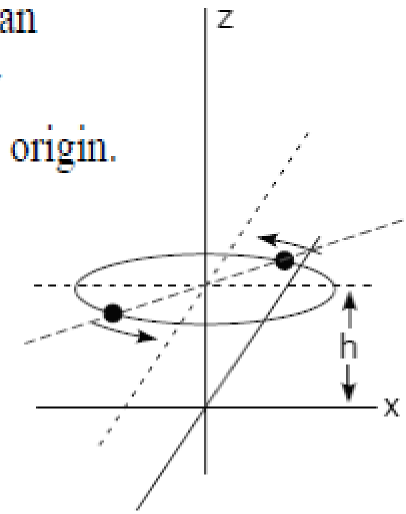
•The *total external torque* acting on the body, which may come from forces acting at many different points.

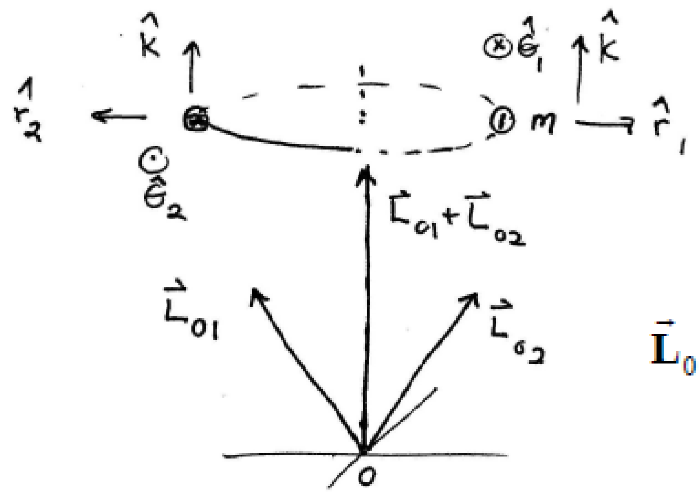
•The particles may not be rigidly connected to each other, they might have relative motion .

In the continuous case, the sums need to be replaced with integrals.

An Example:

Two identical particles of mass m move in a circle of radius r, 180° out of phase at an angular speed ω about the z axis in a plane parallel to but a distance h above the x-y plane. Find the magnitude and the direction of the angular momentum $\vec{\mathbf{L}}_0$ relative to the origin.





$$\vec{\mathbf{L}}_0 = \vec{\mathbf{L}}_{0,1} + \vec{\mathbf{L}}_{0,2} = 2mr^2\omega\hat{\mathbf{k}}\,.$$

For explicit calculation:

$$\vec{\mathbf{r}}_{0,1} = r\hat{\mathbf{r}}_1 + h\hat{\mathbf{k}} \quad \vec{\mathbf{v}}_1 = r\omega\,\hat{\boldsymbol{\theta}}_1 \quad \vec{\mathbf{r}}_{0,2} = r\hat{\mathbf{r}}_2 + h\hat{\mathbf{k}} \quad \vec{\mathbf{v}}_2 = r\omega\,\hat{\boldsymbol{\theta}}_2 \quad \hat{\mathbf{r}}_1 = -\hat{\mathbf{r}}_2 \text{ and } \hat{\boldsymbol{\theta}}_1 = -\hat{\boldsymbol{\theta}}_2$$

$$\hat{\mathbf{r}}_1 \times \hat{\boldsymbol{\theta}}_1 = \hat{\mathbf{k}}$$

$$\hat{\mathbf{r}}_2 \times \hat{\boldsymbol{\theta}}_2 = \hat{\mathbf{k}}$$

$$\begin{aligned}
\vec{\mathbf{L}}_0 &= \vec{\mathbf{L}}_{0,1} + \vec{\mathbf{L}}_{0,2} = \vec{\mathbf{r}}_{0,1} \times m\vec{\mathbf{v}}_1 + \vec{\mathbf{r}}_{0,2} \times m\vec{\mathbf{v}}_2 \\
&= (r\hat{\mathbf{r}}_1 + h\hat{\mathbf{k}}) \times mr\omega\,\hat{\boldsymbol{\theta}}_1 + (r\hat{\mathbf{r}}_2 + h\hat{\mathbf{k}}) \times mr\omega\,\hat{\boldsymbol{\theta}}_2 \\
&= 2mr^2\omega\hat{\mathbf{k}} + hmr\omega(-\hat{\mathbf{r}}_1 - \hat{\mathbf{r}}_2) = 2mr^2\omega\hat{\mathbf{k}} + hmr\omega(-\hat{\mathbf{r}}_1 + \hat{\mathbf{r}}_1) = 2mr^2\omega\hat{\mathbf{k}}
\end{aligned}$$

Since the two objects are symmetrically distributed with respect to the z-axis, the angular momentum about any point along the z-axis has the same value.

Conservation of *L* for a system of particles about a Point:

$$\vec{\tau}^{ext} = \frac{d\vec{L}}{dt}, \text{ where } \vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i = \sum_{i=1}^{N} \vec{L}_i \text{ and } \vec{\tau}^{ext} = \sum_{i=1}^{N} \vec{\tau}_i^{ext}$$

If $\vec{\tau}^{ext} = 0,$ the net torque i.e. the sum of torque on individual particles is zero, the total angular momentum $\vec{L}$, the sum of angular momentum of individual particles will be constant.

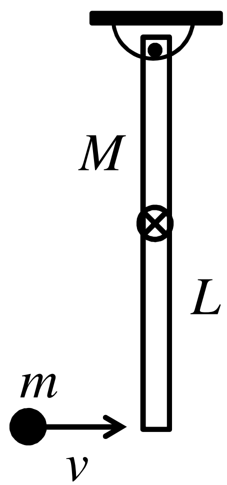
That is

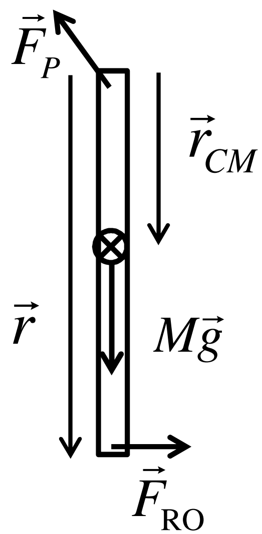
$$\vec{L}_{initial} = \vec{L}_{final}$$

$$\sum_{i=1}^{N} \left(\vec{r}_i \times \vec{p}_i \right)_{initial} = \sum_{i=1}^{N} \left(\vec{r}_i \times \vec{p}_i \right)_{final}$$

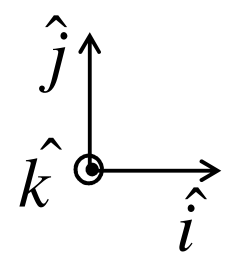
An example:

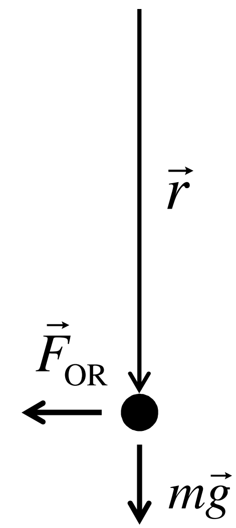
An object of mass *m* and speed *v* strikes a rigid uniform rod of length *L* and mass *M* that is hanging by a frictionless pivot from the ceiling. Immediately after striking the rod, the object continues forward but its speed decreases to *v/2*. Gravity acts with acceleration *g* downward. Find the angular velocity of the rod just after the collision.





Forces on the rod





Forces on the object

$$\vec{\tau}^{ext} = \left[\vec{0}\times\vec{F}_P + \vec{r}_{CM}\times M\vec{g} + \vec{r}\times\vec{F}_{\mathrm{RO}}\right]_{rod} + \left[\vec{r}\times\vec{F}_{\mathrm{OR}} + \vec{r}\times m\vec{g}\right]_{\mathrm{object}}. \quad \text{Since } \vec{F}_{\mathrm{RO}} = -\vec{F}_{\mathrm{OR}},\ \vec{\tau}^{ext} = 0.$$

So, $\vec{L}$ is conserved. $\vec{L}_i = \vec{r}\times m\vec{v} = Lmv\hat{k}$, $\vec{L}_f = \vec{r}\times\frac{1}{2}m\vec{v} + I\omega\hat{k} = \left(\frac{Lmv}{2} + \frac{ML^2\omega}{3}\right)\hat{k}$

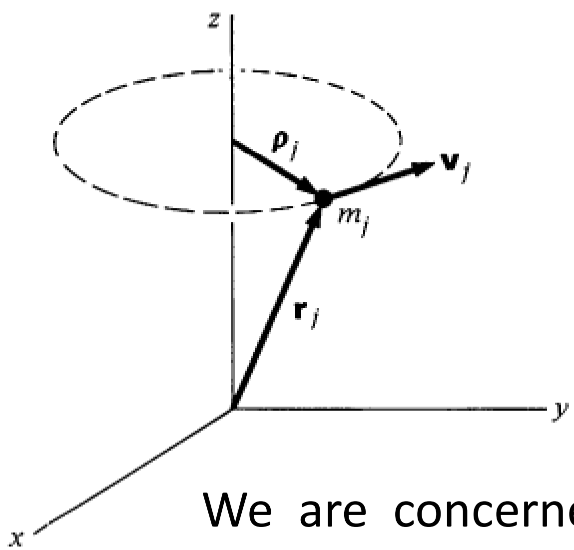
$$Lmv = \frac{Lmv}{2} + \frac{ML^2\omega}{3}, \quad \Rightarrow \quad \omega = \frac{3mv}{2ML}$$

Note: Linear momentum is not conserved.

Angular momentum in Fixed Axis Rotation

By fixed axis we mean that the direction of the axis of rotation is always along the same line; the axis itself may translate.

For example, a car wheel attached to an axle undergoes fixed axis rotation as long as the car drives straight ahead. If the car turns, the wheel must rotate about a vertical axis while simultaneously spinning on the axle; the motion is no longer fixed axis rotation.



$$\vec{L}_j = \vec{r}_j \times \vec{p}_j ,$$

$$\vec{r}_j = \rho_j \hat{\rho} + z_j \hat{k}, \quad \vec{p}_j = m_j \vec{v}_j ,$$

$$\vec{v}_j = \vec{\omega} \times \vec{r}_j = \omega \hat{k} \times \left(\rho_j \hat{\rho} + z_j \hat{k} \right) = \omega \rho_j \hat{\theta}$$

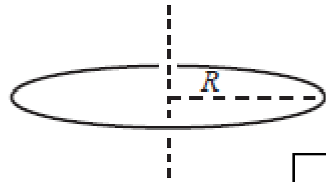
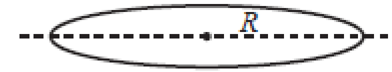
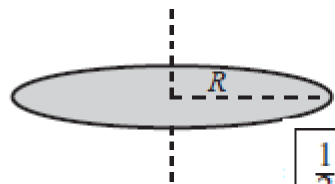
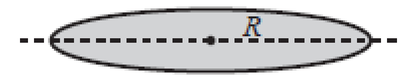
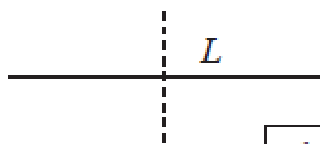
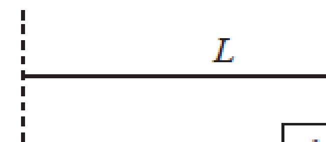
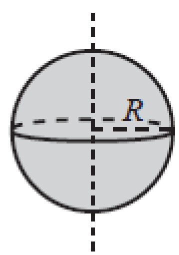
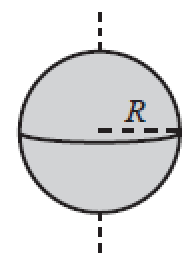
$$\vec{L}_j = \left(\rho_j \hat{\rho} + z_j \hat{k} \right) \times m_j \omega \rho_j \hat{\theta} = m_j \rho_j^2 \omega \hat{k} - m_j z_j \rho_j \omega \hat{\rho}$$

We are concerned only with ***L***$_z$, the component of angular momentum along the axis of rotation.

$$\left(\vec{L}_j \right)_z = L_{jz} = m_j \rho_j^2 \omega, \quad L_z = \sum_j L_{jz} = \left(\sum_j m_j \rho_j^2 \right) \omega = I \omega, \quad I = \sum_j m_j \rho_j^2$$

For continuously distributed mass: $\sum_j m_j \rho_j^2 \longrightarrow \int \rho^2 \, dm,$ *I*=moment of inertia

Moments of inertia of few symmetric objects:

A ring of mass M and radius R, axis through center, perpendicular to plane.	 MR^2	A ring of mass M and radius R, axis through center, in plane.	 $\frac{1}{2}MR^2$
A disk of mass M and radius R, axis through center, perpendicular to plane.	 $\frac{1}{2}MR^2$	A disk of mass M and radius R, axis through center, in plane.	 $\frac{1}{4}MR^2$
A thin uniform rod of mass M and length L, axis through center, perpendicular to rod.	 $\frac{1}{12}ML^2$	A thin uniform rod of mass M and length L axis through end, perpendicular to rod.	 $\frac{1}{3}ML^2$
A spherical shell of mass M and radius R, any axis through center.	 $\frac{2}{3}MR^2$	A solid sphere of mass M and radius R, any axis through center.	 $\frac{2}{5}MR^2$

The parallel-axis theorem:

$$I_z = MR^2 + I_z^{\mathrm{CM}}$$

The perpendicular-axis theorem:

$$I_z = I_x + I_y$$

Dynamics of Pure Rotation about an Axis

Consider fixed axis rotation with no translation of the axis. For instance, the motion of a door on its hinges or the spinning of a fan blade. Motion like this, where there is an axis of rotation at rest, is called pure rotation.

Consider a body rotating with angular velocity ω about the *z* axis. The *z* component of angular momentum is

$$L_z = I\omega.$$

Since $\tau = d\mathbf{L}/dt$, where τ is the external torque,

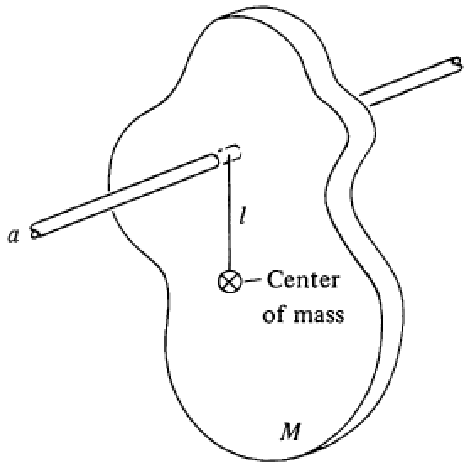
$$\tau_z = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt} = I\alpha,$$

where $\alpha = d\omega/dt$ is called the *angular acceleration*.

$\tau = I\alpha$ is the analog of $\mathbf{F} = m\mathbf{a}$

Note: If the external torque acting on a system is zero, then the angular momentum of the system is constant. Therefore τ=0, consequently α=0; no angular acceleration. The motion corresponds to uniform circular motion with a constant angular velocity.

The Physical Pendulum



$$\tau = I\alpha. \qquad -lW \sin\phi = I_a \ddot{\phi}.$$

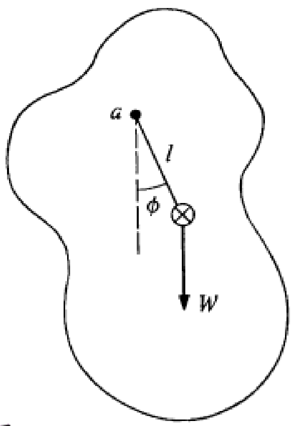
Making the small angle approximation,

$$I_a \ddot{\phi} + Mlg\phi = 0.$$

$$\phi = A\cos\omega t + B\sin\omega t, \qquad \text{where } \omega = \sqrt{Mlg/I_a}.$$

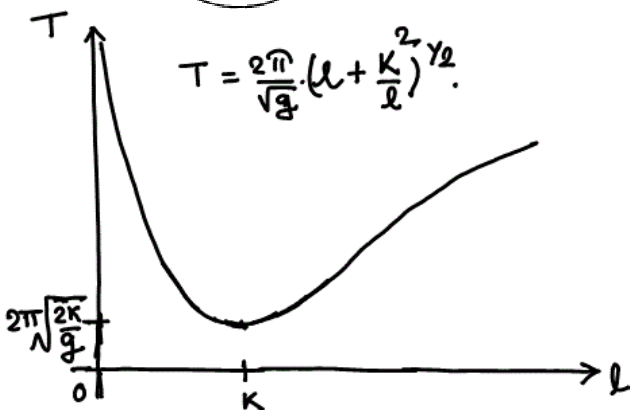
By the parallel axis theorem we have $I_a = I_0 + Ml^2$
$= M(k^2 + l^2)$,

$$k = \sqrt{\frac{I_0}{M}} \qquad I_0 = Mk^2.$$



so that $\omega = \sqrt{\dfrac{gl}{k^2 + l^2}}$.

The simple pendulum corresponds to $k = 0$, $\omega = \sqrt{g/l}$,



$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{(k^2 + l^2)}{gl}}$$

For a bar pendulum: $k = \sqrt{\dfrac{L^2 + b^2}{12}}$

Angular Impulse and Change in Angular Momentum

If there is a total applied torque $\vec{\boldsymbol{\tau}}_S$ about a point S over an interval of time $\Delta t = t_f - t_0$, then the torque applies an *angular impulse* about a point S, given by

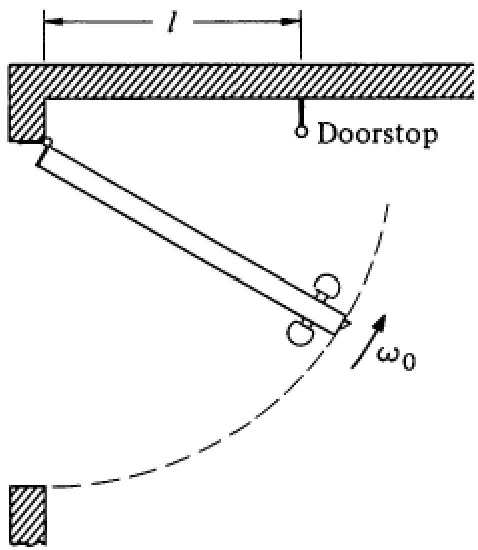
$$\vec{\mathbf{J}}_S = \int_{t_0}^{t_f} \vec{\boldsymbol{\tau}}_S \, dt \, .$$

Because $\vec{\boldsymbol{\tau}}_S = d\vec{\mathbf{L}}_S^{\text{total}} / dt$, the angular impulse about S is equal to the change in angular momentum about S,

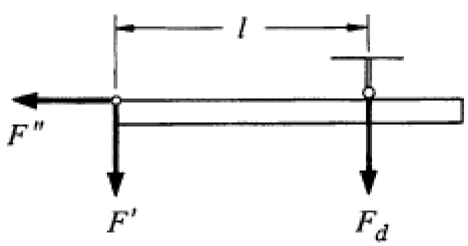
$$\vec{\mathbf{J}}_S = \int_{t_0}^{t_f} \vec{\boldsymbol{\tau}}_S \, dt = \int_{t_0}^{t_f} \frac{d\vec{\mathbf{L}}_S}{dt} dt = \Delta \vec{\mathbf{L}}_S = \vec{\mathbf{L}}_{S,f} - \vec{\mathbf{L}}_{S,0} \, .$$

The Doorstop

The banging of a door against its stop can tear loose the hinges. However, by the proper choice of *l*, the impact forces on the hinge can be made to vanish.



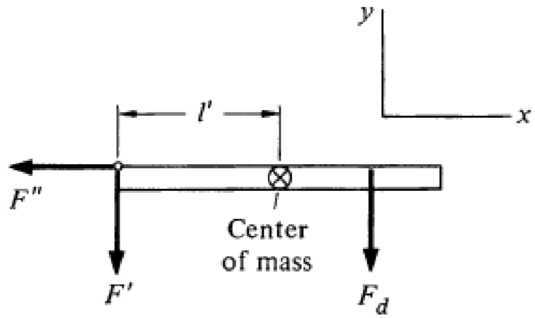
The forces on the door during impact are F_d, due to the stop, and F' and F'' due to the hinge. F'' is the small radial force which provides the centripetal acceleration of the swinging door. F' and F_d are the large impact forces which bring the door to rest when it bangs against the stop. The force on the hinges is equal and opposite to F' and F'' To minimize the stress on the hinges, we must make F' as small as possible.



$$L_{\text{final}} - L_{\text{initial}} = \int_{t_i}^{t_f} \tau \, dt.$$

$$L_{\text{initial}} = I\omega_0, \quad L_{\text{final}} = 0. \qquad \tau = -lF_d,$$

$$I\omega_0 = l \int F_d \, dt, \qquad \text{or} \qquad \int F_d \, dt = I\omega_0/l,$$



The center of mass motion obeys

$$\mathbf{P}_{\text{final}} - \mathbf{P}_{\text{initial}} = \int \mathbf{F}\, dt, \text{ where } \mathbf{F} \text{ is the total force.}$$

The momentum in the y direction

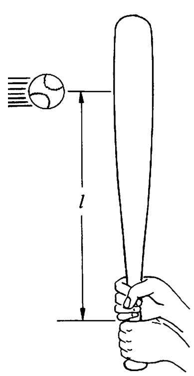
$$\mathbf{P}_{\text{initial}} = MV_y = Ml'\omega_0, \quad P_{\text{final}} = 0, \quad F_y = -(F' + F_d).$$

$$Ml'\omega_0 = \int (F' + F_d)\, dt. \qquad \int F'\, dt = \left(Ml' - \frac{I}{l}\right)\omega_0.$$

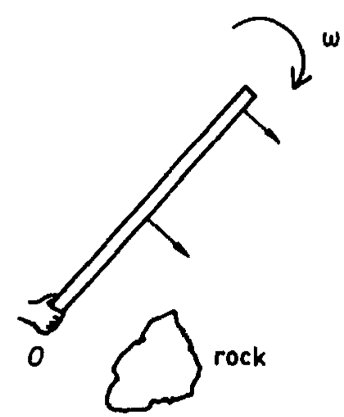
By choosing $l = \dfrac{I}{Ml'}$, the impact force is made zero.

If the door is uniform, and of width w, then $I = Mw^2/3$ and $l' = w/2$.

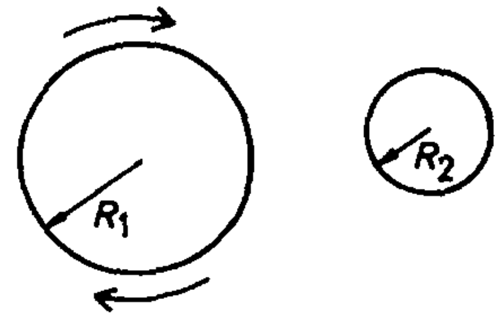
In this case $l = \frac{2}{3}w$.



The distance *l* specified above is called the center of percussion. In batting a baseball or hitting a rock by a rod, it is important to hit the ball at the bat's or rod's center of percussion to avoid a reaction on the hands and a painful sting.



Another Example: The two flywheels are on parallel frictionless shafts but initially do not touch. Say, the larger wheel of radius R_1 and moment of inertia I_1 about its axis of rotation has angular velocity ω_0 about its shaft and the smaller wheel of radius R_2 and moment of inertia I_2 about its axis of rotation is at rest. If the two parallel shafts are moved until contact occurs, find the angular velocity ω of the second wheel after equilibrium occurs (i.e. no further sliding at the point of contact).



Suppose the interacting force between the two wheels is f. If ω' is the angular velocity of the larger wheel after equilibrium,

$$fR_1 = I_1\alpha_1 \quad \text{and} \quad fR_2 = -I_2\alpha_2$$

$$R_1\int f dt = I_1\int_{\omega_0}^{\omega'} d\omega = I_1(\omega' - \omega_0), \quad R_2\int f dt = -I_2\int_0^{\omega} d\omega = -I_2\omega$$

$$\text{No slipping: } \omega' R_1 = \omega R_2$$

$$\frac{I_1}{R_1}\left(\omega_0 - \omega\frac{R_2}{R_1}\right) = \frac{I_2\omega}{R_2} \quad \Rightarrow \quad \omega = \frac{I_1R_1R_2\omega_0}{I_1R_2^2 + I_2R_1^2}$$