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- Convergence of series: $\sum_{n=1}^{\infty} x_n$ is convergent if (s_n) is convergent.

  Otherwise $\sum_{n=1}^{\infty} x_n$ is divergent (not convergent).

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  3. The series $1 - 1 + 1 - 1 + \cdots$ is not convergent.

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- Example: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Cauchy criterion: $\sum_{n=1}^{\infty} x_n$ is convergent iff for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_{n+1} + \cdots + x_m| < \varepsilon$ for all $m > n \geq n_0$.

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Examples: The following series are not convergent.

(a) $\sum_{n=1}^{\infty} \frac{n^2+1}{(n+3)(n+4)}$ (b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

Comparison test: Let (x_n) and (y_n) be sequences in $\mathbb{R}$ such that for some $n_0 \in \mathbb{N}$, $0 \leq x_n \leq y_n$ for all $n \geq n_0$.

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(a) $\sum_{n=1}^{\infty} y_n$ is convergent $\Rightarrow \sum_{n=1}^{\infty} x_n$ is convergent.

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Limit comparison test: Let (x_n) and (y_n) be sequences of positive real numbers such that $\frac{x_n}{y_n} \to \ell \in \mathbb{R}$.

(a) If $\ell \neq 0$, then $\sum_{n=1}^{\infty} x_n$ is convergent iff $\sum_{n=1}^{\infty} y_n$ is convergent.

(b) If $\ell = 0$, then $\sum_{n=1}^{\infty} y_n$ is convergent $\Rightarrow \sum_{n=1}^{\infty} x_n$ is convergent.

Examples: (a) $\sum_{n=1}^{\infty} \frac{1+\sin n}{1+n^2}$ (b) $\sum_{n=1}^{\infty} \frac{1}{2^n+n}$ (c) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$

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Cauchy's condensation test: Let (x_n) be a decreasing sequence of nonnegative real numbers. Then $\sum_{n=1}^{\infty} x_n$ is convergent iff $\sum_{n=1}^{\infty} 2^n x_{2^n}$ is convergent.

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Leibniz's test: Let (x_n) be a decreasing sequence of positive real numbers such that $x_n \to 0$.

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ is convergent

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$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots = \frac{3}{2}s$$

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