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Standard examples: (a) (α^n), where $|\alpha| < 1$
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Definition: (x_n) is increasing if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$.

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Lower bound and infimum (greatest lower bound) are defined similarly.

Result: An increasing sequence (x_n) which is bounded above converges to $\sup\{x_n : n \in \mathbb{N}\}$.

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Result: A sequence in $\mathbb{R}$ is convergent iff it is a Cauchy sequence.

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Subsequence: Let (x_n) be a sequence in $\mathbb{R}$. If (n_k) is a sequence of positive integers such that $n_1 < n_2 < n_3 < \cdots$, then (x_{n_k}) is called a subsequence of (x_n).

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Remark: Let (x_n) be a sequence such that $x_{2n} \to \ell$ and $x_{2n-1} \to \ell$. Then $x_n \to \ell$.

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Similarly, if $x \in \mathbb{R}$, then there exists a sequence (t_n) of irrationals converging to x.