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 - 4. Sequence defined recursively: (x_n) , where $x_1 = 4$ and $x_{n+1} = 2x_n 5$ for all $n \in \mathbb{N}$

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▶ We say: ℓ is a limit of (x_n) : $\lim_{n\to\infty} x_n = \ell$ or $x_n \to \ell$.

Result: The limit of a convergent sequence is unique.

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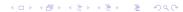
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So, Not bounded implies Not convergent.

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- (a) $x_n + y_n \rightarrow x + y$.
- (b) $\alpha x_n \to \alpha x$ for all $\alpha \in \mathbb{R}$.
- (c) $|x_n| \rightarrow |x|$.
- (d) $x_n y_n \to xy$.
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Standard examples: (a) (α^n) , where $|\alpha| < 1$

(b)
$$(\alpha^{\frac{1}{n}})$$
, where $\alpha > 0$ (c) $(n^{\frac{1}{n}})$

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Examples: (a)
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 (x_n) is decreasing if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.

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Lower bound and infimum (greatest lower bound) are defined similarly.



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Example: Let $x_1 = 1$, $x_{n+1} = \frac{1}{3}(x_n + 1)$ for all $n \in \mathbb{N}$. Then (x_n) is convergent and $\lim_{n \to \infty} x_n = \frac{1}{2}$.

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Cauchy sequence: A sequence (x_n) is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \ge n_0$.

Result: An increasing sequence (x_n) which is bounded above converges to $\sup\{x_n : n \in \mathbb{N}\}.$

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Result: A sequence in $\mathbb R$ is convergent iff it is a Cauchy sequence.



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 for all $n \in \mathbb{N}$,

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Subsequence: Let (x_n) be a sequence in \mathbb{R} . If (n_k) is a sequence of positive integers such that $n_1 < n_2 < n_3 < \cdots$, then (x_{n_k}) is called a subsequence of (x_n) .

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Example: Let $x_n = (-1)^n (1 - \frac{1}{n})$ for all $n \in \mathbb{N}$. Then $x_n \not\to 1$. In fact, (x_n) is not convergent.

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Similarly, if $x \in \mathbb{R}$, then there exists a sequence (t_n) of irrationals converging to x.