MA 101 (Mathematics I)

Hints/Solutions for Practice Problem Set - 2

Ex.1(a) State TRUE or FALSE giving proper justification: If (x_n) is a sequence in $\mathbb{R}$ which converges to 0, then the sequence (x_n^n) must converge to 0.
Solution: The given statement is TRUE. If $x_n \to 0$, then there exists $n_0 \in \mathbb{N}$ such that $|x_n| < \frac{1}{2}$ for all $n \geq n_0$ and so $0 \leq |x_n^n| < (\frac{1}{2})^n$ for all $n \geq n_0$. Since $(\frac{1}{2})^n \to 0$, by sandwich theorem, it follows that $|x_n^n| \to 0$ and consequently $x_n^n \to 0$.

Ex.1(b) State TRUE or FALSE giving proper justification: There exists a non-convergent sequence (x_n) in $\mathbb{R}$ such that the sequence $(x_n + \frac{1}{n}x_n)$ is convergent.
Solution: The given statement is FALSE. If possible, let there exist a non-convergent sequence (x_n) such that the sequence (y_n) is convergent, where $y_n = x_n + \frac{1}{n}x_n = (1+\frac{1}{n})x_n$ for all $n \in \mathbb{N}$. Then, since $x_n = \frac{y_n}{1+\frac{1}{n}}$ for all $n \in \mathbb{N}$ and since $(1+\frac{1}{n})$ converges to $1 \neq 0$, it follows that (x_n) must be convergent, which is a contradiction.

Ex.1(c) State TRUE or FALSE giving proper justification: There exists a non-convergent sequence (x_n) in $\mathbb{R}$ such that the sequence $(x_n^2 + \frac{1}{n}x_n)$ is convergent.
Solution: The given statement is TRUE, because if $x_n = (-1)^n$ for all $n \in \mathbb{N}$, then (x_n) is not convergent, but $(x_n^2 + \frac{1}{n}x_n) = (1 + \frac{(-1)^n}{n})$ is convergent (with limit 1), since $\frac{(-1)^n}{n} \to 0$.

Ex.1(d) State TRUE or FALSE giving proper justification: If (x_n) is a sequence of positive real numbers such that the sequence $((-1)^n x_n)$ converges to $\ell \in \mathbb{R}$, then ℓ must be equal to 0.
Solution: The given statement is TRUE. Since $(-1)^n x_n \to \ell$, the subsequences $((-1)^{2n}x_{2n}) = (x_{2n})$ and $((-1)^{2n-1}x_{2n-1}) = (-x_{2n-1})$ of $((-1)^n x_n)$ must also converge to ℓ. Since $x_{2n} > 0$ for all $n \in \mathbb{N}$, $\ell \geq 0$ and since $-x_{2n-1} < 0$ for all $n \in \mathbb{N}$, $\ell \leq 0$. Hence $\ell = 0$.

Ex.1(e) State TRUE or FALSE giving proper justification: If an increasing sequence (x_n) in $\mathbb{R}$ has a convergent subsequence, then (x_n) must be convergent.
Solution: The given statement is TRUE. Let (x_{n_k}) be a convergent subsequence of (x_n). Then (x_{n_k}) is bounded above, *i.e.* there exists $M > 0$ such that $x_{n_k} \leq M$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, $k \leq n_k$ and since (x_n) is increasing, we get $x_k \leq x_{n_k} \leq M$. Thus (x_n) is bounded above and consequently (x_n) is convergent.

Ex.1(f) State TRUE or FALSE giving proper justification: If (x_n) is a sequence of positive real numbers such that $\lim\limits_{n\to\infty}(n^{\frac{3}{2}}x_n) = \frac{3}{2}$, then the series $\sum\limits_{n=1}^{\infty} x_n$ must be convergent.
Solution: The given statement is TRUE. Since the sequence $(n^{\frac{3}{2}}x_n)$ is convergent, it is bounded and so there exists $M > 0$ such that $0 \leq n^{\frac{3}{2}}x_n \leq M$ for all $n \in \mathbb{N}$. Hence $0 \leq x_n \leq \frac{M}{n^{3/2}}$ for all $n \in \mathbb{N}$. Since $\sum\limits_{n=1}^{\infty} \frac{M}{n^{3/2}}$ is convergent, by comparison test, $\sum\limits_{n=1}^{\infty} x_n$ is convergent.

Ex.1(g) State TRUE or FALSE giving proper justification: If (x_n) is a sequence of positive real numbers such that the series $\sum\limits_{n=1}^{\infty} n^2 x_n^2$ converges, then the series $\sum\limits_{n=1}^{\infty} x_n$ must converge.
Solution: For each $n \in \mathbb{N}$, we have $\sum\limits_{k=1}^{n} x_k = \sum\limits_{k=1}^{n} \frac{1}{k} \cdot kx_k \leq (\sum\limits_{k=1}^{n} \frac{1}{k^2})^{\frac{1}{2}} (\sum\limits_{k=1}^{n} k^2 x_k^2)^{\frac{1}{2}}$ (using Cauchy-Schwarz inequality). Since both the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty} n^2 x_n^2$ are convergent, their sequences of partial sums are bounded. Hence the sequence $\left(\sum\limits_{k=1}^{n} x_k\right)_{n=1}^{\infty}$ of partial sums of the series $\sum\limits_{n=1}^{\infty} x_n$ is

bounded above. Therefore by monotonic criterion for series, the series $\sum_{n=1}^{\infty} x_n$ is convergent.

Ex.1(h) State TRUE or FALSE giving proper justification: If (x_n) is a sequence in $\mathbb{R}$ such that the series $\sum_{n=1}^{\infty} x_n^3$ is convergent, then the series $\sum_{n=1}^{\infty} x_n^4$ must be convergent.

Solution: The given statement is FALSE. If $x_n = \frac{(-1)^n}{n^{1/4}}$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} x_n^3 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$ is convergent by Leibniz's test (we note that the sequence $(\frac{1}{n^{3/4}})$ is decreasing and converges to 0), but $\sum_{n=1}^{\infty} x_n^4 = \sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Ex.1(i) State TRUE or FALSE giving proper justification: If (x_n) is a sequence of positive real numbers such that the series $\sum_{n=1}^{\infty} x_n^3$ is convergent, then the series $\sum_{n=1}^{\infty} x_n^4$ must be convergent.

Solution: The given statement is TRUE. If $\sum_{n=1}^{\infty} x_n^3$ is convergent, then $x_n^3 \to 0$. So there exists $n_0 \in \mathbb{N}$ such that $x_n^3 < 1$ for all $n \geq n_0$. Hence $x_n < 1$ for all $n \geq n_0$ and therefore $0 < x_n^4 < x_n^3$ for all $n \geq n_0$. Since $\sum_{n=1}^{\infty} x_n^3$ is convergent, by comparison test, $\sum_{n=1}^{\infty} x_n^4$ must be convergent.

Ex.1(j) State TRUE or FALSE giving proper justification: If (x_n) is a sequence of positive real numbers such that the series $\sum_{n=1}^{\infty} x_n^4$ is convergent, then the series $\sum_{n=1}^{\infty} x_n^3$ must be convergent.

Solution: The given statement is FALSE. If $x_n = \frac{1}{n^{1/3}}$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} x_n^4 = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ is convergent, but $\sum_{n=1}^{\infty} x_n^3 = \sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Ex.1(k) State TRUE or FALSE giving proper justification: If $f : \mathbb{R} \to \mathbb{R}$ is continuous at both 2 and 4, then f must be continuous at some $c \in (2, 4)$.

Solution: The given statement is FALSE. Let $f(x) = \begin{cases} (x-2)(x-4) & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Let (x_n) be any sequence in $\mathbb{R}$ such that $x_n \to 2$. Since $|f(x_n)| \leq |(x_n - 2)(x_n - 4)| \to 0$, $f(x_n) \to 0 = f(2)$. This shows that $f : \mathbb{R} \to \mathbb{R}$ is continuous at 2. Similarly f is continuous at 4. Let $c \in (2, 4)$. Then there exist sequences (r_n) in $\mathbb{Q}$ and (t_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $r_n \to c$ and $t_n \to c$. Since $f(r_n) = (r_n - 2)(r_n - 4) \to (c-2)(c-4) \neq 0$ and since $f(t_n) \to 0$, it follows that f cannot be continuous at c.

Ex.1(l) State TRUE or FALSE giving proper justification: There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$ and $f(x) \in \mathbb{R} \setminus \mathbb{Q}$ for all $x \in \mathbb{Q}$.

Solution: The given statement is FALSE. If possible, let there exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \in \mathbb{Q}$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$ and $f(x) \in \mathbb{R} \setminus \mathbb{Q}$ for all $x \in \mathbb{Q}$. Let $g(x) = x - f(x)$ for all $x \in \mathbb{R}$. Then $g : \mathbb{R} \to \mathbb{R}$ is continuous and $g(x) \in \mathbb{R} \setminus \mathbb{Q}$ for all $x \in \mathbb{R}$. By the intermediate value theorem, it follows that g must be a constant function. Hence $g(x) = g(0)$ for all $x \in \mathbb{R}$ and so $f(x) = x + f(0)$ for all $x \in \mathbb{R}$. In particular, we get $f(f(0)) = 2f(0)$, which is a contradiction, since $f(0) = -g(0) \in \mathbb{R} \setminus \mathbb{Q}$.

Ex.1(m) State TRUE or FALSE giving proper justification: If $f : [1, 2] \to \mathbb{R}$ is a differentiable function, then the derivative f' must be bounded on $[1, 2]$.

Solution: The given statement is FALSE. Let $f(x) = \begin{cases} (x-1)^2 \sin \frac{1}{(x-1)^2} & \text{if } 1 < x \leq 2, \\ 0 & \text{if } x = 1. \end{cases}$

Clearly $f : [1, 2] \to \mathbb{R}$ is differentiable on $(1, 2]$ with $f'(x) = 2(x-1) \sin \frac{1}{(x-1)^2} - \frac{2}{x-1} \cos \frac{1}{(x-1)^2}$ for all $x \in (1, 2]$. Also, since $\left| \frac{f(x) - f(1)}{x-1} \right| = |x - 1| |\sin \frac{1}{(x-1)^2}| \leq |x - 1|$ for all $x \in (1, 2]$, it follows that

$\lim\limits_{x\to 1+} \frac{f(x)-f(1)}{x-1} = 0$ and hence f is differentiable at 1 (with $f'(1) = 0$). If $x_n = 1 + \frac{1}{\sqrt{2n\pi}}$ for all $n \in \mathbb{N}$, then $x_n \in [1, 2]$ for all $n \in \mathbb{N}$ and $f'(x_n) = -2\sqrt{2n\pi} \to -\infty$, which shows that f' is not bounded on $[1, 2]$.

Ex.1(n) State TRUE or FALSE giving proper justification: If $f : [0, \infty) \to \mathbb{R}$ is differentiable such that $f(0) = 0 = \lim\limits_{x\to\infty} f(x)$, then there must exist $c \in (0, \infty)$ such that $f'(c) = 0$.
Solution: The given statement is TRUE. If possible, let $f'(x) \neq 0$ for all $x \in (0, \infty)$. Then by the intermediate value property of derivatives, either $f'(x) > 0$ for all $x \in (0, \infty)$ or $f'(x) < 0$ for all $x \in (0, \infty)$. We assume that $f'(x) > 0$ for all $x \in (0, \infty)$. (The other case is almost similar.) Then f is strictly increasing on $[0, \infty)$ and so $f(x) > f(1) > f(0) = 0$ for all $x \in (1, \infty)$. This contradicts the given fact that $\lim\limits_{x\to\infty} f(x) = 0$. Hence there exists $c \in (0, \infty)$ such that $f'(c) = 0$.

Ex.1(o) State TRUE or FALSE giving proper justification: If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then for each $c \in \mathbb{R}$, there must exist $a, b \in \mathbb{R}$ with $a < c < b$ such that $f(b) - f(a) = (b - a)f'(c)$.
Solution: The given statement is FALSE. Let $f(x) = x^3$ for all $x \in \mathbb{R}$, so that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. If possible, let there exist $a, b \in \mathbb{R}$ with $a < 0 < b$ such that $f(b) - f(a) = (b - a)f'(0)$. Then $b^3 - a^3 = (b - a) \cdot 0 = 0 \Rightarrow b^3 = a^3$, which is not true, since $a < 0$ and $b > 0$.

Ex.1(p) State TRUE or FALSE giving proper justification: The function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x + \sin x$ for all $x \in \mathbb{R}$, is strictly increasing on $\mathbb{R}$.
Solution: The given statement is TRUE. Since $f'(x) = 1 + \cos x \geq 0$ for all $x \in \mathbb{R}$, f is increasing on $\mathbb{R}$. If possible, let there exist $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$ such that $f(x_1) = f(x_2)$. Then f must be constant on $[x_1, x_2]$ and so $f'(x) = 0$ for all $x \in [x_1, x_2]$. This implies that $\cos x = -1$ for all $x \in [x_1, x_2]$, which is not true. Therefore f is strictly increasing on $\mathbb{R}$.

Ex.1(r) State TRUE or FALSE giving proper justification: If $f : [0, 1] \to \mathbb{R}$ is a bounded function such that $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f(\frac{k}{n})$ exists (in $\mathbb{R}$), then f must be Riemann integrable on $[0, 1]$.
Solution: The given statement is FALSE. If $f(x) = \begin{cases} 0 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 1 & \text{if } x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}), \end{cases}$
then $f : [0, 1] \to \mathbb{R}$ is a bounded function and we know that f is not Riemann integrable on $[0, 1]$. However, since $f(\frac{k}{n}) = 0$ for $k = 1, ..., n$ and for all $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} f(\frac{k}{n}) = 0$.

Ex.2(a) For all $n \in \mathbb{N}$, let $a_n = n + \frac{1}{n}$ and $x_n = \frac{1}{n^2}(a_1 + \cdots + a_n)$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: For all $n \in \mathbb{N}$, $x_n = \frac{1}{n^2}[(1+2+\cdots+n)+(1+\frac{1}{2}+\cdots+\frac{1}{n})] = \frac{1}{2}(1+\frac{1}{n})+\frac{1}{n}\cdot\frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{n}$. Since $\frac{1}{n} \to 0$, by the solution of Ex.4 of Practice Problem Set - 2, we get $\frac{1}{n}(1+\frac{1}{2}+\cdots+\frac{1}{n}) \to 0$. It follows (by limit rules for algebraic operations) that (x_n) is convergent with limit $\frac{1}{2}(1 + 0) + 0.0 = \frac{1}{2}$.

Alternative solution: We can show that $\lim\limits_{n\to\infty} \frac{1}{n^2}(1 + \frac{1}{2} + \cdots + \frac{1}{n}) = 0$ even without using Ex.4 of Practice Problem Set - 2. We have $0 \leq \frac{1}{n^2}(1 + \frac{1}{2} + \cdots + \frac{1}{n}) \leq \frac{1}{n^2}(1 + \cdots + 1) = \frac{1}{n}$ for all $n \in \mathbb{N}$. Since $\frac{1}{n} \to 0$, by sandwich theorem, it follows that $\frac{1}{n^2}(1 + \frac{1}{2} + \cdots + \frac{1}{n}) \to 0$.

Ex.2(b) Let $x_n = (n^2 + 1)^{\frac{1}{8}} - (n + 1)^{\frac{1}{4}}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Hint: We have $x_n = (n^2 + 1)^{\frac{1}{8}} - (n^2)^{\frac{1}{8}} + n^{\frac{1}{4}} - (n + 1)^{\frac{1}{4}}$ for all $n \in \mathbb{N}$. Now consider the first two terms together and the last two terms together. The limit is 0.

Ex.2(c) Let $x_n = (n^2 + n)^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $1 \leq x_n \leq (2n^2)^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. Since $2^{\frac{1}{n}} \to 1$ and $n^{\frac{1}{n}} \to 1$, it follows that

$(2n^2)^{\frac{1}{n}} = 2^{\frac{1}{n}}(n^{\frac{1}{n}})^2 \to 1$. Hence by sandwich theorem, (x_n) is convergent with limit 1.

Ex.2(d) Let $x_n = 5^n(\frac{1}{n^3} - \frac{1}{n!})$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: Let $a_n = \frac{5^n}{n^3}$ and $b_n = \frac{5^n}{n!}$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim\limits_{n\to\infty} \frac{5}{(1+\frac{1}{n})^3} = 5 > 1$ and $\lim\limits_{n\to\infty} |\frac{b_{n+1}}{b_n}| = \lim\limits_{n\to\infty} \frac{5}{n+1} = 0 < 1$, the sequence (a_n) is not convergent and the sequence (b_n) is convergent (with limit 0). Since $(x_n) = (a_n) - (b_n)$, it follows that (See Ex.1(c) of Practice Problem Set - 1) (x_n) is not convergent.

Ex.2(e) Let $x_n = \frac{1}{1.n} + \frac{1}{2.(n-1)} + \frac{1}{3.(n-2)} + \cdots + \frac{1}{n.1}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $x_n = \frac{1}{n+1}[(1+\frac{1}{n}) + (\frac{1}{2} + \frac{1}{n-1}) + \cdots + (\frac{1}{n}+1)] = \frac{2n}{n+1} \cdot \frac{1}{n}(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ for all $n \in \mathbb{N}$. Since $\frac{1}{n} \to 0$, $\frac{1}{n}(1+\frac{1}{2}+\cdots+\frac{1}{n}) \to 0$ (using the solution of Ex.4 of Practice Problem Set - 2) and $\frac{2n}{n+1} = \frac{2}{1+\frac{1}{n}} \to 2$. Hence by limit rule for product, (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = 0$.

Ex.2(f) Let $x_n = \frac{n}{3} - [\frac{n}{3}]$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $x_{3n} = 0$ and $x_{3n+1} = \frac{1}{3}$ for all $n \in \mathbb{N}$. Thus (x_n) has two subsequences (x_{3n}) and (x_{3n+1}) converging to two different limits, *viz.* 0 and $\frac{1}{3}$ respectively. Therefore (x_n) is not convergent.

Ex.2(g) Let $x_1 = 1$ and $x_{n+1} = (\frac{n}{n+1})x_n^2$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: Clearly $x_n \geq 0$ for all $n \in \mathbb{N}$. Also, we have $x_1 = 1$ and if we assume that $x_k \leq 1$ for some $k \in \mathbb{N}$, then $x_{k+1} = (\frac{k}{k+1})x_k^2 \leq 1$. Hence by the principle of mathematical induction, $x_n \leq 1$ for all $n \in \mathbb{N}$. This gives $x_{n+1} = (\frac{n}{n+1}x_n)x_n \leq x_n$ for all $n \in \mathbb{N}$. Thus (x_n) is decreasing and bounded below and hence (x_n) is convergent. If $\ell = \lim\limits_{n\to\infty} x_n$, then we have $\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \frac{n}{n+1}(\lim\limits_{n\to\infty} x_n)^2 \Rightarrow \ell = \ell^2 \Rightarrow \ell = 0$ or 1. Since $\ell = \inf\{x_n : n \in \mathbb{N}\} \leq x_2 = \frac{1}{2}$, we must have $\ell = 0$.

Ex.2(h) Let $a, b \in \mathbb{R}$, $x_1 = a$, $x_2 = b$ and $x_{n+2} = \frac{1}{2}(x_n + x_{n+1})$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution: We have $x_{n+1} - x_n = (-\frac{1}{2})(x_n - x_{n-1}) = \cdots = (-\frac{1}{2})^{n-1}(x_2 - x_1)$ for all $n \in \mathbb{N}$. Hence $x_n = x_1 + (x_n - x_{n-1}) + \cdots + (x_2 - x_1) = a + [(-\frac{1}{2})^{n-2} + \cdots + 1](x_2 - x_1) = a + \frac{2}{3}[1 - (-\frac{1}{2})^{n-1}](b-a)$ for all $n \in \mathbb{N}$. Since $(-\frac{1}{2})^n \to 0$, (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = a + \frac{2}{3}(1-0)(b-a) = \frac{1}{3}(a+2b)$.

Alternative solution: The convergence of (x_n) can also be shown as follows.
We have $x_{n+2} - x_{n+1} = (-\frac{1}{2})(x_{n+1} - x_n)$ for all $n \in \mathbb{N}$, so that $|x_{n+2} - x_{n+1}| = \frac{1}{2}|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$. Hence it follows that (x_n) is a Cauchy sequence in $\mathbb{R}$ and therefore (x_n) converges.

Ex.2(i) Let $0 < x_n < 1$ and $x_n(1 - x_{n+1}) > \frac{1}{4}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find the limit if it is convergent.
Solution Using the *A.M.* > *G.M.* inequality, we have $\frac{x_n + (1-x_{n+1})}{2} \geq \sqrt{x_n(1-x_{n+1})} > \frac{1}{2}$ for all $n \in \mathbb{N}$. Hence $x_n > x_{n+1}$ for all $n \in \mathbb{N}$ and so (x_n) is decreasing. Since $x_n > 0$ for all $n \in \mathbb{N}$, (x_n) is bounded below. Therefore (x_n) is convergent. If $\lim\limits_{n\to\infty} x_n = \ell$, then $\lim\limits_{n\to\infty} x_{n+1} = \ell$. Since $x_n(1 - x_{n+1}) > \frac{1}{4}$ for all $n \in \mathbb{N}$, we get $\ell(1-\ell) \geq \frac{1}{4} \Rightarrow (2\ell - 1)^2 \leq 0 \Rightarrow (2\ell-1)^2 = 0 \Rightarrow \ell = \frac{1}{2}$.

Ex.3 Let (x_n) be any non-constant sequence in $\mathbb{R}$ such that $x_{n+1} = \frac{1}{2}(x_n + x_{n+2})$ for all $n \in \mathbb{N}$. Show that (x_n) cannot converge.
Solution: For each $n \in \mathbb{N}$, $2x_{n+1} = x_n + x_{n+2} \Rightarrow x_{n+2} - x_{n+1} = x_{n+1} - x_n$. If $d = x_2 - x_1$, then $x_n = x_1 + (n-1)d$ for all $n \in \mathbb{N}$. Since (x_n) is not a constant sequence, $d \neq 0$. Given any $M > 0$,

choosing $n \in \mathbb{N}$ satisfying $n > 1 + \frac{M+|x_1|}{|d|}$, we find that $|x_n| > M$. Thus (x_n) is unbounded and consequently (x_n) cannot converge.

Ex.4 Let (x_n) be a sequence in $\mathbb{R}$ and let $y_n = \frac{1}{n}(x_1 + \cdots + x_n)$ for all $n \in \mathbb{N}$. If (x_n) is convergent, then show that (y_n) is also convergent.
If (y_n) is convergent, is it necessary that (x_n) is (i) convergent? (ii) bounded?
Solution: Let $x_n \to \ell \in \mathbb{R}$ and let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|x_n - \ell| < \frac{\varepsilon}{2}$ for all $n > N$. Now for all $n > N$, we have $|y_n - \ell| = \frac{1}{n}|(x_1-\ell)+\cdots+(x_n-\ell)| \le \frac{1}{n}\sum_{i=1}^{N}|x_i-\ell| + \frac{1}{n}\sum_{i=N+1}^{n}|x_i-\ell|$.
We choose $K \in \mathbb{N}$ such that $\frac{1}{K}\sum_{i=1}^{N}|x_i-\ell| < \frac{\varepsilon}{2}$. Let $n_0 = \max\{N, K\}$. Then $n_0 \in \mathbb{N}$ and for all $n > n_0$, we have $|y_n - \ell| < \frac{\varepsilon}{2} + (\frac{n-N}{n})\frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence (y_n) is convergent (with limit ℓ).
If (y_n) is convergent, then it is not necessary that (x_n) is convergent. For example, let (x_n) be the sequence $(1, -1, 1, -1, ...)$, which is not convergent. But since $|y_n| \le \frac{1}{n}$ for all $n \in \mathbb{N}$, we see that $y_n \to 0$.
If (y_n) is convergent, then it is not even necessary that (x_n) is bounded. For example, let (x_n) be the sequence $(1, -1, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}, ...)$, which is not bounded. But $y_{2n} = 0$ and $y_{2n-1} = \frac{\sqrt{n}}{2n-1}$ for all $n \in \mathbb{N}$, so that $|y_n| \le \frac{\sqrt{n+1}}{\sqrt{2n}} = \frac{1}{\sqrt{2}}\sqrt{\frac{1}{n} + \frac{1}{n^2}} \to 0$. Hence $y_n \to 0$.

Ex.5 If (x_n) is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty}(x_{n+1} - x_n) = 5$, then determine $\lim_{n\to\infty}\frac{x_n}{n}$.
Solution: Let $y_n = x_{n+1} - x_n$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} y_n = 5$, by the solution of Ex.4 of Practice Problem Set - 2, we have $\lim_{n\to\infty}\frac{1}{n}(y_1+\cdots+y_n) = 5$. Since $y_1+\cdots+y_n = (x_2-x_1)+\cdots+(x_{n+1}-x_n) = x_{n+1} - x_1$ for all $n \in \mathbb{N}$, we get $\lim_{n\to\infty}\frac{x_{n+1}-x_1}{n} = 5$. Now $\frac{x_{n+1}}{n+1} = \frac{x_{n+1}-x_1}{n}\cdot\frac{n}{n+1} + \frac{x_1}{n+1}$ for all $n \in \mathbb{N}$ and hence by applying the limit rules, we obtain $\lim_{n\to\infty}\frac{x_{n+1}}{n+1} = 5.1 + 0 = 5$. It follows that $\lim_{n\to\infty}\frac{x_n}{n} = 5$.

Ex.6 If $x_1 = \frac{3}{4}$ and $x_{n+1} = x_n - x_n^{n+1}$ for all $n \in \mathbb{N}$, then examine whether the sequence (x_n) is convergent.
Solution: We have $0 < x_1 < 1$ and if we assume that $0 < x_k < 1$ for some $k \in \mathbb{N}$, then $0 < x_{k+1} = x_k(1 - x_k^k) < 1$. Hence by the principle of mathematical induction $0 < x_n < 1$ for all $n \in \mathbb{N}$. Also, $x_{n+1} = x_n(1 - x_n^n) < x_n$ (since $1 - x_n^n < 1$ and $x_n > 0$) for all $n \in \mathbb{N}$. Thus the sequence (x_n) is decreasing and bounded below and so it is convergent.

Ex.7 Let $a > 0$ and let $x_1 = 0$, $x_{n+1} = x_n^2 + a$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) is convergent iff $a \le \frac{1}{4}$.
Solution: If (x_n) is convergent, then there exists $\ell \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = \ell$. Since $x_{n+1} = x_n^2 + a$ for all $n \in \mathbb{N}$, we get $\lim_{n\to\infty} x_{n+1} = (\lim_{n\to\infty} x_n)^2 + a$, which gives $\ell^2 - \ell + a = 0$. Since $\ell \in \mathbb{R}$, we must have $1 - 4a \ge 0$, *i.e.* $a \le \frac{1}{4}$.
Conversely, let $a \le \frac{1}{4}$. We note that $x_1 = 0$ and $x_{n+1} = x_n^2 + a \ge 0$ for all $n \in \mathbb{N}$. Now $x_2 = a > x_1$ and if we assume that $x_{k+1} > x_k$ for some $k \in \mathbb{N}$, then $x_{k+2} = x_{k+1}^2 + a > x_k^2 + a = x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} > x_n$ for all $n \in \mathbb{N}$. Also, $x_1 < \frac{1}{2}$ and if $x_k < \frac{1}{2}$ for some $k \in \mathbb{N}$, then $x_{k+1} \le x_k^2 + \frac{1}{4} < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Hence by the principle of mathematical induction, $x_n < \frac{1}{2}$ for all $n \in \mathbb{N}$. Thus (x_n) is increasing and bounded above and therefore (x_n) is convergent.

Ex.8 For $a \in \mathbb{R}$, let $x_1 = a$ and $x_{n+1} = \frac{1}{4}(x_n^2 + 3)$ for all $n \in \mathbb{N}$. Examine the convergence of the sequence (x_n) for different values of a. Also, find $\lim_{n\to\infty} x_n$ whenever it exists.
Solution: If $\ell = \lim_{n\to\infty} x_n$ exists (in $\mathbb{R}$), then the only possible values of ℓ are 1 and 3 (since $\ell = \frac{1}{4}(\ell^2 + 3)$, *i.e.* $(\ell - 1)(\ell - 3) = 0$). We have $x_n > 0$ and $x_{n+2} - x_{n+1} = \frac{1}{4}(x_{n+1}^2 - x_n^2)$ for all $n \in \mathbb{N}$. Also $x_2 - x_1 = \frac{1}{4}(a - 1)(a - 3)$.
Let $a > 3$. Then $x_2 > x_1$ and if we assume that $x_{k+1} > x_k$ for some $k \in \mathbb{N}$, then from above, we get

$x_{k+2} > x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} > x_n$ for all $n \in \mathbb{N}$. It follows that (x_n) cannot converge. (Because if (x_n) converges, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\} \geq x_1 > 3$, which is not possible as we have seen above that the only possible values of $\lim_{n\to\infty} x_n$ are 1 and 3.)
If $a = 3$, then $x_n = 3$ for all $n \in \mathbb{N}$, and hence (x_n) converges to 3.
Let $1 < a < 3$. Then $x_2 < x_1$ and if we assume that $x_{k+1} < x_k$ for some $k \in \mathbb{N}$, then from above, we get $x_{k+2} < x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Also, by the principle of mathematical induction, we can show that in this case $x_n > 1$ for all $n \in \mathbb{N}$. (Because $x_{n+1} - 1 = \frac{1}{4}(x_n^2 - 1)$ for all $n \in \mathbb{N}$ and $x_1 > 1$.) Hence (x_n) converges to 1. ($x_n \not\to 3$ because $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\} \leq x_1 < 3$.)
Let $0 \leq a \leq 1$. Then $x_2 \geq x_1$ and if we assume that $x_{k+1} \geq x_k$ for some $k \in \mathbb{N}$, then from above, we get $x_{k+2} \geq x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. Also, by the principle of mathematical induction, we can show that in this case $x_n \leq 1$ for all $n \in \mathbb{N}$. (Because $x_{n+1} - 1 = \frac{1}{4}(x_n^2 - 1)$ for all $n \in \mathbb{N}$ and $x_1 \leq 1$.) Hence (x_n) converges to 1. (Since $x_n \leq 1$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} x_n \neq 3$.)
The case for $a < 0$ is treated by considering $-a$ in place of a, because x_2 is same irrespective of whether we choose $x_1 = a$ or $x_1 = -a$. Hence we can say that for $-1 \leq a < 0$, $x_n \to 1$, for $-3 < a < -1$, $x_n \to 1$, for $a = -3$, $x_n \to 3$ and for $a < -3$, (x_n) does not converge.

Ex.9 If $x_n = (1 + \frac{1}{n})^n$ and $y_n = (1 + \frac{1}{n})^{n+1}$ for all $n \in \mathbb{N}$, then show that the sequence (x_n) is increasing, the sequence (y_n) is decreasing and both (x_n) and (y_n) are bounded.
Solution: For each $n \in \mathbb{N}$, applying the $A.M. \geq G.M.$ inequality for the numbers $a_1 = 1$, $a_2 = a_3 = \cdots = a_{n+1} = 1 + \frac{1}{n}$, we get $\frac{1+n(1+\frac{1}{n})}{n+1} \geq (1 + \frac{1}{n})^{\frac{n}{n+1}}$. From this, we get $(1 + \frac{1}{n+1})^{n+1} \geq (1 + \frac{1}{n})^n$ for all $n \in \mathbb{N}$. Therefore the sequence (x_n) is increasing.
Again, for each $n \in \mathbb{N}$, applying $A.M. \geq G.M.$ inequality for the numbers $a_1 = \cdots = a_{n+1} = \frac{n}{n+1}$, $a_{n+2} = 1$, we get $\frac{(n+1)\frac{n}{n+1}+1}{n+2} \geq (\frac{n}{n+1})^{\frac{n+1}{n+2}}$. From this, we get $(1 + \frac{1}{n+1})^{n+2} \leq (1 + \frac{1}{n})^{n+1}$ for all $n \in \mathbb{N}$. Therefore the sequence (y_n) is decreasing.
It is now clear that $0 < x_n \leq (1 + \frac{1}{n})^n(1 + \frac{1}{n}) = y_n \leq y_1 = 4$ for all $n \in \mathbb{N}$ and so both (x_n) and (y_n) are bounded.

Alternative solution: The boundedness of (x_n) can also be proved as follows.
For all $n \in \mathbb{N}$, we have $0 < x_n = 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \cdots + \frac{1}{n^n} \leq 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{2^2}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots + \frac{1}{2^n} \leq 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} = 2 + (1 - \frac{1}{2^n}) < 3$.

Ex.10 Let (x_n) be a sequence in $\mathbb{R}$. If for every $\varepsilon > 0$, there exists a convergent sequence (y_n) in $\mathbb{R}$ such that $|x_n - y_n| < \varepsilon$ for all $n \in \mathbb{N}$, then show that (x_n) is convergent.
Solution: Let $\varepsilon > 0$. Then there exists a convergent sequence (y_n) in $\mathbb{R}$ such that $|x_n - y_n| < \frac{\varepsilon}{3}$ for all $n \in \mathbb{N}$. Since (y_n) is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $|y_n - y_m| < \frac{\varepsilon}{3}$ for all $n, m \geq n_0$. Hence for all $n, m \geq n_0$, we have $|x_n - x_m| \leq |x_n - y_n| + |y_n - y_m| + |y_m - x_m| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Thus (x_n) is a Cauchy sequence in $\mathbb{R}$ and therefore (x_n) is convergent.

Ex.11 Let (x_n) be a sequence in $\mathbb{R}$. Which of the following conditions ensure(s) that (x_n) is a Cauchy sequence (and hence convergent)?

(a) $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$.
(b) $|x_{n+1} - x_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$.
(c) $|x_{n+1} - x_n| \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

Solution: Let $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $|x_{n+1} - x_n| = \frac{1}{n+1} < \frac{1}{n}$ for all $n \in \mathbb{N}$ and so $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$. Thus both the conditions (a) and (b) are satisfied for the sequence (x_n). However, (x_n) is not a Cauchy sequence, since we know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent and so its sequence of partial sums, which is (x_n), is not a Cauchy sequence.

Now, let (x_n) be a sequence in $\mathbb{R}$ such that $|x_{n+1} - x_n| \le \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by Cauchy's criterion for convergence of series, there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(m-1)^2} < \varepsilon$ for all $m > n > n_0$. Hence for all $m > n > n_0$, we get $|x_m - x_n| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + \cdots + x_{m-1} - x_m| \le |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \le \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(m-1)^2} < \varepsilon$. Therefore (x_n) is a Cauchy sequence.

Ex.12 Let (x_n) be a sequence in $\mathbb{R}$ such that each of the subsequences (x_{2n}), (x_{2n-1}) and (x_{3n}) converges. Show that (x_n) is convergent.
Solution: Let $x_{2n} \to x$, $x_{2n-1} \to y$ and $x_{3n} \to z$, where $x, y, z \in \mathbb{R}$. Clearly (x_{6n}) is a subsequence of each of the sequences (x_{2n}) and (x_{3n}). So $x_{6n} \to x$ and $x_{6n} \to z$. This implies that $x = z$. Again, $(x_{3(2n-1)})$ is a subsequence of each of the sequences (x_{2n-1}) and (x_{3n}). So $x_{3(2n-1)} \to y$ and $x_{3(2n-1)} \to z$. This implies that $y = z$. Thus each of the subsequences (x_{2n}) and (x_{2n-1}) of (x_n) converges to the same limit $x = y$. Therefore it follows that (x_n) is convergent (with limit $x = y$).

Ex.13(a) Examine whether the series $\sum\limits_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$ is convergent.
Solution: We have $(\log n)^{\log n} = (e^{\log(\log n)})^{\log n} = (e^{\log n})^{\log(\log n)} = n^{\log(\log n)}$ for all $n \ge 2$. Also, $\log(\log n) > 2$ for all $n > e^{e^2}$. We choose $n_0 \in \mathbb{N}$ such that $n_0 > e^{e^2}$. Then $\frac{1}{(\log n)^{\log n}} = \frac{1}{n^{\log(\log n)}} \le \frac{1}{n^2}$ for all $n \ge n_0$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by comparison test, the given series is convergent.

Ex.13(b) Examine whether the series $\sum\limits_{n=1}^{\infty} \frac{2^n - n}{n^2}$ is convergent.
Solution: Since $\lim\limits_{n\to\infty} \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} = 2 > 1$, the sequence $(\frac{2^n}{n^2})$ is not convergent. Also, since $\frac{1}{n} \to 0$, the sequence $(\frac{2^n - n}{n^2})$ is not convergent (being the difference of a divergent and a convergent sequence). Hence the given series is not convergent.

Ex.13(c) Examine whether the series $\sum\limits_{n=1}^{\infty} \frac{\frac{1}{2} + (-1)^n}{n}$ is convergent.
Solution: We know that the series $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$ is divergent. Also, since $(\frac{1}{n})$ is a decreasing sequence of positive real numbers with $\frac{1}{n} \to 0$, by Leibniz's test, the series $\frac{(-1)^n}{n}$ is convergent. Since the given series is the sum of the divergent series $\sum\limits_{n=1}^{\infty} \frac{1}{2n}$ and the convergent series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$, it is not convergent.

Ex.13(d) Examine whether the series $\frac{1}{\sqrt{1}} - \frac{1}{2} + \frac{1}{\sqrt{3}} - \frac{1}{4} + \frac{1}{\sqrt{5}} - \frac{1}{6} + \cdots$ is convergent.
Solution: For each $n \in \mathbb{N}$, let s_n denote the nth partial sum of the given series. Since $\frac{1}{\sqrt{2n-1}} - \frac{1}{2n} \ge \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}$ for all $n \in \mathbb{N}$, we get $s_{2n} = \frac{1}{\sqrt{1}} - \frac{1}{2} + \frac{1}{\sqrt{3}} - \frac{1}{4} + \cdots + \frac{1}{\sqrt{2n-1}} - \frac{1}{2n} \ge \frac{1}{2}(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ for all $n \in \mathbb{N}$. Again, the sequence $(1 + \frac{1}{2} + \cdots + \frac{1}{n})$ of partial sums of the divergent series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is not bounded above and hence the sequence (s_n) is not bounded above. Thus the sequence (s_n) is not convergent and consequently the given series is not convergent.

Ex.13(e) Examine whether the series $1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + x^6 + 2x^7 + \cdots$ is convergent, where $x \in \mathbb{R}$.
Solution: Taking the given series as $\sum\limits_{n=1}^{\infty} a_n$, we have $a_{2n} = 2x^{2n-1}$ and $a_{2n-1} = x^{2n-2}$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} |a_{2n}|^{\frac{1}{2n}} = |x| = \lim\limits_{n\to\infty} |a_{2n-1}|^{\frac{1}{2n-1}}$, we get $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = |x|$. Hence by the root test, the given series is absolutely convergent (and hence convergent) if $|x| < 1$ and is not convergent if $|x| > 1$. If $|x| = 1$, then $\lim\limits_{n\to\infty} |a_{2n}| = \lim\limits_{n\to\infty} 2|x|^{2n-1} = 2 \ne 0$ and so $a_n \not\to 0$. Consequently the given

series is not convergent if $|x| = 1$.

Ex.14 If (x_n) is a sequence in $\mathbb{R}$ such that $\lim\limits_{n\to\infty} x_n = 0$, then show that the series $\sum\limits_{n=1}^{\infty} \frac{x_n}{x_n^2+n^2}$ is absolutely convergent.
Solution: Since $\lim\limits_{n\to\infty} x_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n| < 1$ for all $n \geq n_0$. Hence for all $n \geq n_0$, $\left|\frac{x_n}{x_n^2+n^2}\right| = \frac{|x_n|}{x_n^2+n^2} \leq \frac{1}{n^2}$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by comparison test, $\sum\limits_{n=1}^{\infty} \left|\frac{x_n}{x_n^2+n^2}\right|$ is convergent. Consequently $\sum\limits_{n=1}^{\infty} \frac{x_n}{x_n^2+n^2}$ is absolutely convergent.

Ex.15 Let the series $\sum\limits_{n=1}^{\infty} x_n$ be convergent, where $x_n > 0$ for all $n \in \mathbb{N}$. Examine whether the following series are convergent.

(a) $\sum\limits_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$

(b) $\sum\limits_{n=1}^{\infty} \frac{x_n+2^n}{x_n+3^n}$

Solution: (a) For all $n \in \mathbb{N}$, $0 \leq (\sqrt{x_n} - \frac{1}{n})^2 = x_n - 2\frac{\sqrt{x_n}}{n} + \frac{1}{n^2}$. Hence $\frac{\sqrt{x_n}}{n} \leq \frac{1}{2}(x_n + \frac{1}{n^2})$ for all $n \in \mathbb{N}$. Since both $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converge, $\sum\limits_{n=1}^{\infty} \frac{1}{2}(x_n + \frac{1}{n^2})$ also converges. Therefore by comparison test, $\sum\limits_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ converges.

(b) Let $a_n = \frac{x_n+2^n}{x_n+3^n}$ and $b_n = (\frac{2}{3})^n$ for all $n \in \mathbb{N}$. Since $\sum\limits_{n=1}^{\infty} x_n$ converges, $x_n \to 0$, and so $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\frac{1}{2^n}x_n+1}{\frac{1}{3^n}x_n+1} = 1$. Since $\sum\limits_{n=1}^{\infty} b_n$ converges, by limit comparison test, $\sum\limits_{n=1}^{\infty} a_n$ also converges.

Alternative solution for (b): Since $\sum\limits_{n=1}^{\infty} x_n$ converges, $x_n \to 0$, and so there exists $n_0 \in \mathbb{N}$ such that $|x_n| < 1$ for all $n \geq n_0$. Hence for all $n \geq n_0$, $\frac{x_n+2^n}{x_n+3^n} < \frac{x_n+2^n}{3^n} < (\frac{1}{3})^n + (\frac{2}{3})^n$. Since both $\sum\limits_{n=1}^{\infty} (\frac{1}{3})^n$ and $\sum\limits_{n=1}^{\infty} (\frac{2}{3})^n$ converge, $\sum\limits_{n=1}^{\infty} [(\frac{1}{3})^n + (\frac{2}{3})^n]$ converges. Hence by comparison test, $\sum\limits_{n=1}^{\infty} \frac{x_n+2^n}{x_n+3^n}$ converges.

Ex.16 If $\sum\limits_{n=1}^{\infty} x_n$ is a convergent series, where $x_n > 0$ for all $n \in \mathbb{N}$, then show that it is possible for the series $\sum\limits_{n=1}^{\infty} \sqrt{\frac{x_n}{n}}$ to converge as well as not to converge.
Hint: If $x_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$, then $\sum\limits_{n=1}^{\infty} x_n$ is convergent and $\sum\limits_{n=1}^{\infty} \sqrt{\frac{x_n}{n}} = \sum\limits_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is also convergent. On the other hand, if $x_1 = 0$ and $x_n = \frac{1}{n(\log n)^2}$ for all $n \geq 2$, then by Cauchy's condensation test, $\sum\limits_{n=1}^{\infty} x_n = \sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ is convergent but $\sum\limits_{n=1}^{\infty} \sqrt{\frac{x_n}{n}} = \sum\limits_{n=2}^{\infty} \frac{1}{n \log n}$ is not convergent.

Ex.17 Let (x_n) be a sequence in $\mathbb{R}$ with $\lim\limits_{n\to\infty} x_n = 0$. Show that there exists a subsequence (x_{n_k}) of (x_n) such that the series $\sum\limits_{k=1}^{\infty} x_{n_k}$ is absolutely convergent.
Solution: Since $\lim\limits_{n\to\infty} x_n = 0$, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $|x_n| < \frac{1}{2^k}$ for all $n \geq n_k$. We can choose (n_k) such that $n_1 < n_2 < \cdots$. Then (x_{n_k}) is a subsequence of (x_n) satisfying $|x_{n_k}| < \frac{1}{2^k}$ for all $k \in \mathbb{N}$. Since $\sum\limits_{k=1}^{\infty} \frac{1}{2^k}$ is convergent, by comparison test, $\sum\limits_{k=1}^{\infty} |x_{n_k}|$ is convergent, *i.e.* $\sum\limits_{k=1}^{\infty} x_{n_k}$ is absolutely convergent.

Ex.18 If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then show that there exist non-negative continuous functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f = g - h$.
Solution: Let $g = \frac{1}{2}(|f| + f)$ and $h = \frac{1}{2}(|f| - f)$. Then both $g, h : \mathbb{R} \to \mathbb{R}$ are non-negative continuous functions and $g - h = f$.

Ex.19 Give an example (with justification) of a function from $\mathbb{R}$ onto $\mathbb{R}$ which is not continuous at any point of $\mathbb{R}$.
Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ x+1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
If $y \in \mathbb{Q}$, then $f(y) = y$ and if $y \in \mathbb{R} \setminus \mathbb{Q}$, then $y - 1 \in \mathbb{R} \setminus \mathbb{Q}$ and $f(y-1) = y$. Hence f is onto. Let $x \in \mathbb{R}$. Then there exist sequences (r_n) in $\mathbb{Q}$ and (t_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $r_n \to x$ and $t_n \to x$. Now $f(r_n) = r_n \to x$ and $f(t_n) = t_n + 1 \to x + 1$. Since $x \neq x + 1$, it follows that f cannot be continuous at x. Since $x \in \mathbb{R}$ was arbitrary, f is not continuous at any point of $\mathbb{R}$.

Ex.20 Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, then show that $f(x) = f(1)x$ for all $x \in \mathbb{R}$.
Solution: If $n \in \mathbb{N}$, then $f(n) = f(1+\cdots+1) = f(1)+\cdots+f(1) = nf(1)$. Also $f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$. If $m = -n$, where $n \in \mathbb{N}$, then $0 = f(0) = f(m + n) = f(m) + f(n) \Rightarrow f(m) = -f(n) = -nf(1) = mf(1)$. If $r \in \mathbb{Q}$, then $r = \frac{m}{n}$ for some $m \in \mathbb{Z}$, $n \in \mathbb{N}$. So $mf(1) = f(m) = f(\frac{m}{n} + \cdots + \frac{m}{n}) = f(\frac{m}{n}) + \cdots + f(\frac{m}{n}) = nf(\frac{m}{n}) \Rightarrow f(\frac{m}{n}) = \frac{m}{n}f(1)$, *i.e.* $f(r) = rf(1)$. Let $x \in \mathbb{R}$. Then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to x$. So $r_n - x \to 0$ and since f is continuous at 0, $0 = f(0) = \lim_{n\to\infty} f(r_n - x) = \lim_{n\to\infty} [f(r_n) - f(x)] = \lim_{n\to\infty} r_n f(1) - f(x) = xf(1) - f(x)$. Consequently $f(x) = f(1)x$.

Ex.21 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(\frac{1}{2}(x+y)) = \frac{1}{2}(f(x)+f(y))$ for all $x, y \in \mathbb{R}$. Show that there exist $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{R}$.
Solution: Let $g(x) = f(x) - f(0)$ for all $x \in \mathbb{R}$. The given condition gives $\frac{1}{2}(f(x) + f(y)) = f(\frac{1}{2}(x+y)) = f(\frac{1}{2}(x+y+0)) = \frac{1}{2}(f(x+y)+f(0))$ for all $x, y \in \mathbb{R}$. So $g(x+y) = f(x+y) - f(0) = f(x) + f(y) - 2f(0) = g(x) + g(y)$ for all $x, y \in \mathbb{R}$. Since f is continuous, $g : \mathbb{R} \to \mathbb{R}$ is also continuous and hence by Ex.20 of Practice Problem Set-2, $g(x) = g(1)x$ for all $x \in \mathbb{R}$. Thus for all $x \in \mathbb{R}$, $f(x) - f(0) = x(f(1) - f(0))$. Taking $a = f(1) - f(0) \in \mathbb{R}$ and $b = f(0) \in \mathbb{R}$, we get $f(x) = ax + b$ for all $x \in \mathbb{R}$.

Ex.22 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that for each $x \in \mathbb{Q}$, $f(x)$ is an integer. If $f(\frac{1}{2}) = 2$, then find $f(\frac{1}{3})$.
Solution: Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to x$. Since f is continuous at x, $f(r_n) \to f(x)$. If $f(x)$ is not an integer, then $f(x) - [f(x)] > 0$ and so there exists $n_0 \in \mathbb{N}$ such that $|f(r_{n_0}) - f(x)| < \frac{1}{2}(f(x) - [f(x)])$, which is not possible, because $f(r_{n_0})$ is an integer (by hypothesis). Therefore $f(x)$ is an integer. Thus $f(x)$ is an integer for each $x \in \mathbb{R}$ and by the intermediate value theorem, $f : \mathbb{R} \to \mathbb{R}$ must be a constant function. Consequently $f(\frac{1}{3}) = f(\frac{1}{2}) = 2$.

Alternative method for showing that $f(x)$ is an integer: The sequence $(f(r_n))$, being convergent, is a Cauchy sequence. Hence there exists $n_0 \in \mathbb{N}$ such that $|f(r_n) - f(r_{n_0})| < \frac{1}{2}$ for all $n \geq n_0$. Since $f(r_n)$ is an integer for each $n \in \mathbb{N}$ (by hypothesis), we must have $f(r_n) = f(r_{n_0})$ for all $n \in \mathbb{N}$. Consequently $f(r_n) \to f(r_{n_0})$ and therefore $f(x) = f(r_{n_0})$, which is an integer.

Ex.23 Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Show that f is a constant function.
Solution: Let $x > 0$. By hypothesis $f(x) = f(x^{1/2}) = f(x^{1/4}) = \cdots = f(x^{1/2^n})$ for all $n \in \mathbb{N}$. Since $x^{1/2^n} \to 1$ (as $(x^{1/2^n})$ is a subsequence of $(x^{1/n})$ and $x^{1/n} \to 1$) and since f is continuous at 1, $f(x^{1/2^n}) \to f(1)$. It follows that $f(x) = f(1)$. Also $f(-x) = f((-x)^2) = f(x^2) = f(x)$.

Hence $f(x) = f(1)$ for all $x(\neq 0) \in \mathbb{R}$. Since f is continuous at 0, $f(0) = \lim_{x\to 0} f(x) = f(1)$. Thus $f(x) = f(1)$ for all $x \in \mathbb{R}$. Consequently $f : \mathbb{R} \to \mathbb{R}$ is a constant function.

Ex.24 If $f : [0, 1] \to \mathbb{R}$ is continuous, then show that
(a) there exist $a, b \in [0, 1]$ such that $a - b = \frac{1}{2}$ and $f(a) - f(b) = \frac{1}{2}(f(1) - f(0))$.
(b) there exist $a, b \in [0, 1]$ such that $a - b = \frac{1}{3}$ and $f(a) - f(b) = \frac{1}{3}(f(1) - f(0))$.

Solution: (a) Let $g(x) = f(x + \frac{1}{2}) - f(x)$ for all $x \in [0, \frac{1}{2}]$. Since f is continuous, $g : [0, \frac{1}{2}] \to \mathbb{R}$ is continuous. If $g(0) = g(\frac{1}{2})$, then $f(\frac{1}{2}) - f(0) = \frac{1}{2}(f(1) - f(0))$ and so we get the result by taking $a = \frac{1}{2}$ and $b = 0$. If $g(0) \neq g(\frac{1}{2})$, then $\frac{1}{2}(f(1) - f(0)) = \frac{1}{2}(g(0) + g(\frac{1}{2}))$ lies (strictly) between $g(0)$ and $g(\frac{1}{2})$. Hence by the intermediate value theorem, there exists $c \in (0, \frac{1}{2})$ such that $g(c) = \frac{1}{2}(f(1) - f(0))$, *i.e.* $f(c+\frac{1}{2}) - f(c) = \frac{1}{2}(f(1) - f(0))$. Taking $a = c + \frac{1}{2}$ and $b = c$, we get the result.

Alternative solution: Let $g(x) = f(x + \frac{1}{2}) - f(x) - \frac{1}{2}(f(1) - f(0))$ for all $x \in [0, \frac{1}{2}]$. Since f is continuous, $g : [0, \frac{1}{2}] \to \mathbb{R}$ is continuous. Also, $g(0) = f(\frac{1}{2}) - \frac{1}{2}f(0) - \frac{1}{2}f(1)$ and $g(\frac{1}{2}) = \frac{1}{2}f(1) - f(\frac{1}{2}) + \frac{1}{2}f(0) = -g(0)$. If $g(0) = 0$, then we get the result by taking $a = \frac{1}{2}$ and $b = 0$. If $g(0) \neq 0$, then $g(\frac{1}{2})$ and $g(0)$ are of opposite signs and hence by the intermediate value theorem, there exists $c \in (0, \frac{1}{2})$ such that $g(c) = 0$, *i.e.* $f(c+\frac{1}{2}) - f(c) = \frac{1}{2}(f(1) - f(0))$. Taking $a = c + \frac{1}{2}$ and $b = c$, we get the result.

(b) Let $g(x) = f(x+\frac{1}{3}) - f(x) - \frac{1}{3}(f(1) - f(0))$ for all $x \in [0, \frac{2}{3}]$. Since $f : [0, 1] \to \mathbb{R}$ is continuous, $g : [0, \frac{2}{3}] \to \mathbb{R}$ is continuous. Also, $g(0) + g(\frac{1}{3}) + g(\frac{2}{3}) = 0$. If at least one of $g(0)$, $g(\frac{1}{3})$ and $g(\frac{2}{3})$ is 0, then the result follows immediately. Otherwise, at least two of $g(0)$, $g(\frac{1}{3})$ and $g(\frac{2}{3})$ are of opposite signs and hence by the intermediate value property of continuous functions, there exists $c \in (0, \frac{2}{3})$ such that $g(c) = 0$, *i.e.* $f(c+\frac{1}{3}) - f(c) = \frac{1}{3}(f(1) - f(0))$. We take $a = c + \frac{1}{3}$ and $b = c$.

Ex.25 Let $f : [a, b] \to \mathbb{R}$ be continuous. For $n \in \mathbb{N}$, let $x_1, ..., x_n \in [a, b]$ and let $\alpha_1, ..., \alpha_n$ be nonzero real numbers having same sign. Show that there exists $c \in [a, b]$ such that
$f(c) \sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \alpha_i f(x_i)$.
(In particular, this shows that if $f : [a, b] \to \mathbb{R}$ is continuous and if for $n \in \mathbb{N}$, $x_1, ..., x_n \in [a, b]$, then there exists $\xi \in [a, b]$ such that $f(\xi) = \frac{1}{n}(f(x_1) + \cdots + f(x_n))$.)

Solution: Let $\alpha = \sum_{i=1}^{n} \alpha_i$. Then $\alpha \neq 0$ and $\frac{\alpha_i}{\alpha} > 0$ for $i = 1, ..., n$. Since $f : [a, b] \to \mathbb{R}$ is continuous, there exist $y, z \in [a, b]$ such that $f(y) \leq f(x) \leq f(z)$ for all $x \in [a, b]$. In particular, $f(y) \leq f(x_i) \leq f(z)$ for $i = 1, ..., n$ and so $\sum_{i=1}^{n} (\frac{\alpha_i}{\alpha}) f(y) \leq \sum_{i=1}^{n} (\frac{\alpha_i}{\alpha}) f(x_i) \leq \sum_{i=1}^{n} (\frac{\alpha_i}{\alpha}) f(z) \Rightarrow f(y) \leq \frac{1}{\alpha} \sum_{i=1}^{n} \alpha_i f(x_i) \leq f(z)$. By the intermediate value theorem, there exists c between y and z (both inclusive) and so $c \in [a, b]$ such that $f(c) = \frac{1}{\alpha} \sum_{i=1}^{n} \alpha_i f(x_i)$, *i.e.* $f(c)\alpha = \sum_{i=1}^{n} \alpha_i f(x_i)$.

(If we take $\alpha_1 = \cdots = \alpha_n = \frac{1}{n}$, then $\sum_{i=1}^{n} \alpha_i = 1$ and so applying the above result, we get the required conclusion.)

Ex.26 Let $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ be continuous such that $\sup\{f(x) : x \in [0, 1]\} = \sup\{g(x) : x \in [0, 1]\}$. Show that there exists $c \in [0, 1]$ such that $f(c) = g(c)$.
Solution: Since $f : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \to \mathbb{R}$ are continuous, there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = \sup\{f(x) : x \in [0, 1]\}$ and $g(x_2) = \sup\{g(x) : x \in [0, 1]\}$. Since $f(x_1) = g(x_2)$ (by hypothesis), we get $f(x_1) \geq g(x_1)$ and $f(x_2) \leq g(x_2)$. If $f(x_1) = g(x_1)$ or $f(x_2) = g(x_2)$, then the result follows immediately. So we may now assume that $f(x_1) > g(x_1)$ and $f(x_2) < g(x_2)$. Let $\varphi(x) = f(x) - g(x)$ for all $x \in [0, 1]$. Since f and g are continuous, $\varphi : [0, 1] \to \mathbb{R}$ is continuous. Also $\varphi(x_1) > 0$ and $\varphi(x_2) < 0$. Hence by the intermediate value theorem, there exists c between

x_1 and x_2 such that $\varphi(c) = 0$, *i.e.* $f(c) = g(c)$.

Ex.27 Let $f : (0, \infty) \to \mathbb{R}$ be continuous such that $\lim\limits_{x\to 0+} f(x) = 0$ and $\lim\limits_{x\to\infty} f(x) = 1$. Show that there exists $c \in (0, \infty)$ such that $f(c) = \frac{\sqrt{3}}{2}$.
Hint: Since $\lim\limits_{x\to 0+} f(x) = 0 < \frac{1}{4}$ and $\lim\limits_{x\to\infty} f(x) = 1 > \frac{9}{10}$, there exist $x_1, x_2 \in (0, \infty)$ with $x_1 < x_2$ such that $f(x_1) < \frac{1}{4}$ and $f(x_2) > \frac{9}{10}$. Since $\frac{1}{4} < \frac{\sqrt{3}}{2} < \frac{9}{10}$, by the intermediate value theorem, there exists $c \in (x_1, x_2)$ such that $f(c) = \frac{\sqrt{3}}{2}$.

Ex.28 Let $f : (a, b) \to \mathbb{R}$ be continuous. If both $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to b-} f(x)$ exist (in $\mathbb{R}$), then show that f is bounded.
Solution: Let $\lim\limits_{x\to a+} f(x) = \ell_1$ and $\lim\limits_{x\to b-} f(x) = \ell_2$, where $\ell_1, \ell_2 \in \mathbb{R}$. Then there exist $\delta_1, \delta_2 > 0$ such that $|f(x) - \ell_1| < 1$ for all $x \in (a, a + \delta_1)$ and $|f(x) - \ell_2| < 1$ for all $x \in (b - \delta_2, b)$. Hence $|f(x)| < 1 + |\ell_1|$ for all $x \in (a, a + \delta_1)$ and $|f(x) < 1 + |\ell_2|$ for all $x \in (b - \delta_2, b)$. Since f is continuous on $[a + \frac{\delta_1}{2}, b - \frac{\delta_2}{2}]$, f is bounded on $[a + \frac{\delta_1}{2}, b - \frac{\delta_2}{2}]$. So there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in [a + \frac{\delta_1}{2}, b - \frac{\delta_2}{2}]$. Choosing $K = \max\{M, 1 + |\ell_1|, 1 + |\ell_2|\} > 0$, we find that $|f(x)| \le K$ for all $x \in (a, b)$. Consequently f is bounded.

Ex.29 Consider the continuous function $f : (0, 1] \to \mathbb{R}$, where $f(x) = 1 - (1 - x)\sin\frac{1}{x}$ for all $x \in (0, 1]$. Does there exist $x_0 \in (0, 1]$ such that $f(x_0) = \sup\{f(x) : x \in (0, 1]\}$? Justify.
Solution: For all $x \in (0, 1]$, we have $f(x) \le 1 + (1 - x) < 2$. Hence 2 is an upper bound of $\{f(x) : x \in (0, 1]\}$. Therefore there exists $u \in \mathbb{R}$ such that $u = \sup\{f(x) : x \in (0, 1]\} \le 2$. Now $\frac{2}{(4n-1)\pi} \in (0, 1]$ for all $n \in \mathbb{N} \Rightarrow u \ge f\left(\frac{2}{(4n-1)\pi}\right) = 2 - \frac{2}{(4n-1)\pi}$ for all $n \in \mathbb{N} \Rightarrow u \ge 2$ (since $\lim\limits_{n\to\infty} \frac{2}{(4n-1)\pi} = 0$). Thus $u = 2$ and so (as seen at the beginning) $f(x) < u$ for all $x \in (0, 1]$, *i.e.* there cannot exist any $x_0 \in (0, 1]$ such that $f(x_0) = \sup\{f(x) : x \in (0, 1]\}$.

Ex.30 Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(a) = f(b)$. Show that for each $\varepsilon > 0$, there exist distinct $x, y \in [a, b]$ such that $|x - y| < \varepsilon$ and $f(x) = f(y)$.
Solution: We first show that there exist $x_1, y_1 \in [a, b]$ such that $|x_1 - y_1| = \frac{1}{2}(b - a)$ and $f(x_1) = f(y_1)$. Let $g(x) = f(x + \frac{b-a}{2}) - f(x)$ for all $x \in [a, \frac{a+b}{2}]$. Since f is continuous, $g : [a, \frac{a+b}{2}] \to \mathbb{R}$ is continuous. Also $g(a) = f(\frac{a+b}{2}) - f(a)$ and $g(\frac{a+b}{2}) = f(b) - f(\frac{a+b}{2}) = -g(a)$, since $f(a) = f(b)$. If $g(a) = 0$, then we can take $x_1 = \frac{a+b}{2}$ and $y_1 = a$. Otherwise, $g(\frac{a+b}{2})$ and $g(a)$ are of opposite signs and hence by the intermediate value theorem, there exists $c \in (a, \frac{a+b}{2})$ such that $g(c) = 0$, *i.e.* $f(c + \frac{b-a}{2}) = f(c)$. We take $x_1 = c + \frac{b-a}{2}$ and $y_1 = c$.
Repeating the same procedure as above we get $x_2, y_2 \in [a, b]$ such that $|x_2 - y_2| = \frac{1}{2}|x_1 - y_1| = \frac{1}{2^2}(b - a)$ and $f(x_2) = f(y_2)$. Continuing in this way, for each $n \in \mathbb{N}$, there exist $x_n, y_n \in [a, b]$ such that $|x_n - y_n| = \frac{1}{2^n}(b - a)$ and $f(x_n) = f(y_n)$. If $\varepsilon > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0}}(b - a) < \varepsilon$. Hence the result follows by choosing $x = x_{n_0}$ and $y = y_{n_0}$.
Alternative solution: By continuity of f on $[a, b]$, there exist $x_0, y_0 \in [a, b]$ such that $f(y_0) \le f(x) \le f(x_0)$ for all $x \in [a, b]$. If both $x_0, y_0 \in \{a, b\}$, then f must be a constant function and so the result is obvious. Hence we assume that $x_0 \in (a, b)$. (The case of $y_0 \in (a, b)$ is similar.) Let $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $(x_0 - \frac{\varepsilon}{n_0}, x_0 + \frac{\varepsilon}{n_0}) \subset [a, b]$. If any two of the three values $f(x_0 - \frac{\varepsilon}{4n_0})$, $f(x_0)$ and $f(x_0 + \frac{\varepsilon}{4n_0})$ are equal, then the result follows immediately. Otherwise, we assume without loss of generality that $f(x_0 - \frac{\varepsilon}{4n_0}) < f(x_0 + \frac{\varepsilon}{4n_0}) < f(x_0)$. By the intermediate value theorem, there exists $c \in (x_0 - \frac{\varepsilon}{4n_0}, x_0)$ such that $f(c) = f(x_0 + \frac{\varepsilon}{4n_0})$. We get the result by taking $x = x_0 + \frac{\varepsilon}{4n_0}$ and $y = c$.

Ex.31 Give an example (with justification) of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable only at 2.

Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} (x-2)^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

We have $\lim\limits_{x \to 2} \frac{f(x)-f(2)}{x-2} = \lim\limits_{x \to 2} \frac{f(x)}{x-2} = 0$, since $\left|\frac{f(x)}{x-2}\right| \leq |x-2|$ for all $x(\neq 2) \in \mathbb{R}$. Hence f is differentiable at 2.

Again, let $x(\neq 2) \in \mathbb{R}$. Then there exist sequences (r_n) in $\mathbb{Q}$ and (t_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $r_n \to x$ and $t_n \to x$. Now $f(r_n) = (r_n - 2)^2 \to (x-2)^2$ and $f(t_n) \to 0$ (since $f(t_n) = 0$ for all $n \in \mathbb{N}$). Since $(x-2)^2 \neq 0$, it follows that f cannot be continuous at x and consequently f cannot be differentiable at x. Therefore f is differentiable only at 2.

Ex.32 Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) - f(y) \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Show that f is a constant function.

Solution: The given condition implies that $|f(x) - f(y)| \leq |x-y|^2$ for all $x, y \in \mathbb{R}$. Let $y \in \mathbb{R}$. Then for all $x(\neq y) \in \mathbb{R}$, we have $|\frac{f(x)-f(y)}{x-y}| \leq |x-y| \Rightarrow \lim\limits_{x \to y} \frac{f(x)-f(y)}{x-y} = 0$, *i.e.* $f'(y) = 0$. Thus $f'(y) = 0$ for all $y \in \mathbb{R}$. Consequently $f : \mathbb{R} \to \mathbb{R}$ is a constant function.

Ex.33 If $m, k \in \mathbb{N}$, then evaluate $\lim\limits_{n \to \infty} \left(\frac{(n+1)^m + (n+2)^m + \cdots + (n+k)^m}{n^{m-1}} - kn \right)$.

Solution: The given limit equals $\lim\limits_{n \to \infty} \sum\limits_{i=1}^{k} i \frac{(1+\frac{i}{n})^m - 1}{\frac{i}{n}} = \sum\limits_{i=1}^{k} i \lim\limits_{n \to \infty} \frac{(1+\frac{i}{n})^m - 1}{\frac{i}{n}} = \sum\limits_{i=1}^{k} i \frac{d}{dx}(1+x)^m|_{x=0}$ (using sequential criterion of limit) $= \frac{k(k+1)}{2} m$.

Ex.34 Let $f : (a,b) \to \mathbb{R}$ and $g : (a,b) \to \mathbb{R}$ be differentiable at $c \in (a,b)$ such that $f(c) = g(c)$ and $f(x) \leq g(x)$ for all $x \in (a,b)$. Show that $f'(c) = g'(c)$.

Solution: The given conditions imply that $\frac{f(x)-f(c)}{x-c} \leq \frac{g(x)-g(c)}{x-c}$ for all $x \in (c,b)$ and $\frac{f(x)-f(c)}{x-c} \geq \frac{g(x)-g(c)}{x-c}$ for all $x \in (a,c)$. Since f is differentiable at c, we get $f'(c) = \lim\limits_{x \to c+} \frac{f(x)-f(c)}{x-c} \leq \lim\limits_{x \to c+} \frac{g(x)-g(c)}{x-c} = g'(c)$ and $f'(c) = \lim\limits_{x \to c-} \frac{f(x)-f(c)}{x-c} \geq \lim\limits_{x \to c-} \frac{g(x)-g(c)}{x-c} = g'(c)$. Consequently $f'(c) = g'(c)$.

Ex.35 Let $f : [0,1] \to \mathbb{R}$ be differentiable such that $f(0) = f(1) = 0$. Show that there exists $c \in (0,1)$ such that $f'(c) = f(c)$.

Solution: Let $g(x) = e^{-x} f(x)$ for all $x \in [0,1]$. Then $g : [0,1] \to \mathbb{R}$ is differentiable and $g'(x) = e^{-x}(f'(x) - f(x))$ for all $x \in [0,1]$. Also, since $g(0) = 0 = g(1)$, by Rolle's theorem, there exists $c \in (0,1)$ such that $g'(c) = 0$. Since $e^{-c} \neq 0$, we get $f'(c) = f(c)$.

Ex.36 Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(0) = 0$ and $f'(x) > f(x)$ for all $x \in \mathbb{R}$. Show that $f(x) > 0$ for all $x > 0$.

Solution: If $g(x) = e^{-x} f(x)$ for all $x \in \mathbb{R}$, then $g : \mathbb{R} \to \mathbb{R}$ is differentiable and $g'(x) = e^{-x}(f'(x) - f(x)) > 0$ for all $x \in \mathbb{R}$. Hence g is strictly increasing on $\mathbb{R}$ and so $g(x) > g(0)$ for all $x > 0$. This implies that $f(x) > 0$ for all $x > 0$.

Ex.37 Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that $f(x) \neq 0$ for all $x \in [a,b]$. Show that there exists $c \in (a,b)$ such that $\frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}$.

Solution: Let $g(x) = (x-a)(x-b)f(x)$ for all $x \in [a,b]$. Since $f : [a,b] \to \mathbb{R}$ is differentiable, $g : [a,b] \to \mathbb{R}$ is differentiable (and hence continuous) and $g'(x) = (x-a)(x-b)f'(x) + (x-b)f(x) + (x-a)f(x)$ for all $x \in [a,b]$. Also, $g(a) = 0 = g(b)$. Therefore by Rolle's theorem, there exists $c \in (a,b)$ such that $g'(c) = 0$, *i.e.* $(c-a)(c-b)f'(c) = -(c-b)f(c) - (c-a)f(c)$. Dividing by $(c-a)(c-b)f(c) \neq 0$, we obtain $\frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}$.

Ex.38 Let $f : [0,1] \to \mathbb{R}$ be differentiable such that $f(0) = 0$ and $f(1) = 1$. Show that there exist $c_1, c_2 \in [0,1]$ with $c_1 \neq c_2$ such that $f'(c_1) + f'(c_2) = 2$.

Solution: By the mean value theorem, there exist $c_1 \in (0, \frac{1}{2})$ and $c_2 \in (\frac{1}{2}, 1)$ such that $f(\frac{1}{2}) - f(0) = \frac{1}{2} f'(c_1)$ and $f(1) - f(\frac{1}{2}) = \frac{1}{2} f'(c_2)$. Hence $c_1, c_2 \in [0,1]$ with $c_1 \neq c_2$ such that $f'(c_1) + f'(c_2) =$

$2[f(1) - f(0)] = 2$.

Ex.39 Show that for each $a \in (0,1)$ and for each $b \in \mathbb{R}$, the equation $a \sin x + b = x$ has a unique root in $\mathbb{R}$.
Solution: Let $a \in (0,1)$, $b \in \mathbb{R}$ and let $f(x) = x - a\sin x - b$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable (and also continuous) and $f'(x) = 1 - a\cos x$ for all $x \in \mathbb{R}$. Since $a \in (0,1)$, $a\cos x \le a < 1$ for all $x \in \mathbb{R}$ and so $f'(x) \neq 0$ for all $x \in \mathbb{R}$. As a consequence of Rolle's theorem, the equation $f(x) = 0$ has at most one root in $\mathbb{R}$. Again, $f(b+1) = 1 - a\sin(b+1) > 0$ (since $a\sin(b+1) \le a < 1$) and $f(b-1) = -1 - a\sin(b-1) < 0$ (since $a\sin(b-1) \ge -a > -1$). Hence by the intermediate value property of continuous functions, the equation $f(x) = 0$ has at least one root in $(b-1, b+1)$. Thus the equation $f(x) = 0$, *i.e.* the equation $a\sin x + b = x$ has a unique root in $\mathbb{R}$.

Ex.40(a) Find the number of (distinct) real roots of the equation $3^x + 4^x = 5^x$.
Solution: If $f(x) = (\frac{3}{5})^x + (\frac{4}{5})^x - 1$ for all $x \in \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'(x) = (\frac{3}{5})^x \log(\frac{3}{5}) + (\frac{4}{5})^x \log(\frac{4}{5}) < 0$ for all $x \in \mathbb{R}$. As a consequence of Rolle's theorem, the equation $f(x) = 0$ has at most one real root and hence the given equation has at most one real root. Clearly 2 is a root of the given equation. Therefore the given equation has exactly one (distinct) real root.

Ex.40(b) Find the number of (distinct) real roots of the equation $x^{13} + 7x^3 - 5 = 0$.
Solution: Let $f(x) = x^{13} + 7x^3 - 5$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) = 13x^{12} + 21x^2 > 0$ for all $x > 0$. As a consequence of Rolle's theorem, the equation $f(x) = 0$ has at most one root in $(0, \infty)$. Also, since $f(0) = -5 < 0$ and $f(1) = 3 > 0$, by the intermediate value property of continuous functions, the equation $f(x) = 0$ has least one root in $(0,1)$. Since $f(x) < 0$ for all $x \le 0$, it follows that the given equation has exactly one (distinct) real root.

Ex.41 Show that for each $n \in \mathbb{N}$, the equation $x^n + x - 1 = 0$ has a unique root in $[0,1]$.
If for each $n \in \mathbb{N}$, x_n denotes this root, then show that the sequence (x_n) converges to 1.
Solution: Let $n \in \mathbb{N}$ and let $f_n(x) = x^n + x - 1$ for all $x \in [0,1]$. Then $f_n : [0,1] \to \mathbb{R}$ is differentiable and $f_n'(x) = nx^{n-1} + 1 > 0$ for all $x \in [0,1]$. This shows that f_n is a strictly increasing function on $[0,1]$ and so the equation $f_n(x) = 0$ can have at most one root in $[0,1]$. Again, since $f_n(0) = -1 < 0$ and $f_n(1) = 1 > 0$, by the intermediate value theorem, the equation $f_n(x) = 0$ has at least one root in $(0,1)$. Thus the equation $f_n(x) = 0$ has a unique root in $[0,1]$, which is denoted by x_n.
For each $n \in \mathbb{N}$, $0 < x_n < 1 \Rightarrow f_{n+1}(x_n) = x_n^{n+1} + x_n - 1 < x_n^n + x_n - 1 = 0 = f_{n+1}(x_{n+1}) \Rightarrow x_n < x_{n+1}$, since as shown above, f_{n+1} is strictly increasing on $[0,1]$. Also $x_n \in (0,1)$ for all $n \in \mathbb{N}$. Thus the sequence (x_n) is increasing and bounded and consequently (x_n) is convergent. If $\ell = \lim\limits_{n\to\infty} x_n$, then $0 \le \ell \le 1$ (since $0 < x_n < 1$ for all $n \in \mathbb{N}$). If possible, let $\ell < 1$. Then there exists $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \frac{1}{2}(1-\ell)$ for all $n \ge n_0$. This gives $0 < x_n^n < (\frac{1+\ell}{2})^n$ for all $n \ge n_0$. Since $0 < \frac{1+\ell}{2} < 1$, $(\frac{1+\ell}{2})^n \to 0$ and so $x_n^n \to 0$. Now $x_n^n + x_n - 1 = 0$ for all $n \in \mathbb{N}$ $\Rightarrow \lim\limits_{n\to\infty}(x_n^n + x_n - 1) = 0 \Rightarrow \ell - 1 = 0 \Rightarrow \ell = 1$, which is a contradiction. Hence $\ell = 1$.

Ex.42 Let $f : (0,1) \to \mathbb{R}$ be differentiable and let $|f'(x)| \le 3$ for all $x \in (0,1)$. Show that the sequence $(f(\frac{1}{n+1}))$ converges.
Solution: For all $m, n \in \mathbb{N}$ with $m \neq n$, by the mean value theorem, there exists c between $\frac{1}{m+1}$ and $\frac{1}{n+1}$ such that $|f(\frac{1}{m+1}) - f(\frac{1}{n+1})| = |f'(c)||\frac{1}{m+1} - \frac{1}{n+1}| \le 3(\frac{1}{m} + \frac{1}{n})$. Thus if $\varepsilon > 0$, then choosing $n_0 \in \mathbb{N}$ such that $n_0 > \frac{6}{\varepsilon}$, we find that $|f(\frac{1}{m+1}) - f(\frac{1}{n+1})| \le \frac{6}{n_0} < \varepsilon$ for all $m, n \ge n_0$. Hence $(f(\frac{1}{n+1}))$ is a Cauchy sequence in $\mathbb{R}$ and therefore $(f(\frac{1}{n+1}))$ converges.

Ex.43 Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $\lim\limits_{x\to\infty} f'(x) = 1$. Show that f is unbounded.

Solution: Since $\lim\limits_{x\to+\infty} f'(x) = 1$, there exists $M > 0$ such that $|f'(x) - 1| < \frac{1}{2}$ for all $x > M$ and so $\frac{1}{2} < f'(x) < \frac{3}{2}$ for all $x > M$. If $g(x) = f(x) - \frac{x}{2}$ for all $x \in \mathbb{R}$, then $g'(x) = f'(x) - \frac{1}{2} > 0$ for all $x > M \Rightarrow g$ is strictly increasing on $[M, \infty) \Rightarrow g(x) > g(M)$ for all $x > M \Rightarrow f(x) > \frac{x}{2} + f(M) - \frac{M}{2}$ for all $x > M \Rightarrow \lim\limits_{x\to\infty} f(x) = \infty \Rightarrow f$ is unbounded.

Ex.44 Let $f : [a, b] \to \mathbb{R}$ be twice differentiable and let $f(a) = f(b) = 0$ and $f(c) > 0$, where $c \in (a, b)$. Show that there exists $\xi \in (a, b)$ such that $f''(\xi) < 0$.
Solution: By the mean value theorem, there exist $x_1 \in (a, c)$ and $x_2 \in (c, b)$ such that $f'(x_1) = \frac{f(c)-f(a)}{c-a} = \frac{f(c)}{c-a}$ and $f'(x_2) = \frac{f(b)-f(c)}{b-c} = -\frac{f(c)}{b-c}$. Again, by the mean value theorem, there exists $\xi \in (x_1, x_2)$ (and so $\xi \in (a, b)$) such that $f''(\xi) = \frac{f'(x_2)-f'(x_1)}{x_2-x_1} = -\frac{(b-a)f(c)}{(x_2-x_1)(b-c)(c-a)} < 0$, since $f(c) > 0$.

Ex.45 If $f : [0, 4] \to \mathbb{R}$ is differentiable, then show that there exists $c \in [0, 4]$ such that $f'(c) = \frac{1}{6}(f'(1) + 2f'(2) + 3f'(3))$.
Solution: Let $f'(\alpha) = \min\{f'(1), f'(2), f'(3)\}$ and $f'(\beta) = \max\{f'(1), f'(2), f'(3)\}$, where $\alpha, \beta \in \{1, 2, 3\}$. Then $f'(\alpha) \le \frac{1}{6}(f'(1) + 2f'(2) + 3f'(3)) \le f'(\beta)$ and hence by the intermediate value property of derivatives, there exists $c \in [0, 4]$ such that $f'(c) = \frac{1}{6}(f'(1) + 2f'(2) + 3f'(3))$.

Ex.46 Let $f(x) = \begin{cases} x & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Examine whether f is Riemann integrable on $[0, 1]$. Also, find $\int\limits_0^1 f$, if it exists (in $\mathbb{R}$).
Solution: Clearly f is bounded on $[0, 1]$. Let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[0, 1]$. Since between any two distinct real numbers, there exist a rational as well as an irrational number, it follows that $M_i = x_i$ and $m_i = 0$ for $i = 1, ..., n$. (Note that M_i cannot be less than x_i, because otherwise we can find a rational number r_i between M_i and x_i and so $f(r_i) = r_i > M_i$, which is not possible.) Hence $L(f, P) = 0$ and $U(f, P) = \sum\limits_{i=1}^{n} x_i(x_i - x_{i-1}) = \sum\limits_{i=1}^{n} x_i^2 - \sum\limits_{i=1}^{n} x_i x_{i-1} \ge \frac{1}{2} \sum\limits_{i=1}^{n} (x_i^2 - x_{i-1}^2)$ (since $x_i^2 + x_{i-1}^2 \ge 2x_i x_{i-1}$ for $i = 1, ..., n$) $= \frac{1}{2}$. Consequently $\overline{\int\limits_0^1} f(x)\,dx \ge \frac{1}{2}$ and $\underline{\int\limits_0^1} f(x)\,dx = 0$.
Since $\overline{\int\limits_0^1} f(x)\,dx \ne \underline{\int\limits_0^1} f(x)\,dx$, f is not Riemann integrable on $[0, 1]$.

Ex.47 If $f : [0, 1] \to \mathbb{R}$ is Riemann integrable, then find $\lim\limits_{n\to\infty} \int\limits_0^1 x^n f(x)\,dx$.
Solution: Since f is Riemann integrable on $[0, 1]$, f is bounded on $[0, 1]$. So there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in [0, 1]$. Now $|\int\limits_0^1 x^n f(x)\,dx| \le \int\limits_0^1 |x^n f(x)|\,dx \le M \int\limits_0^1 x^n\,dx = \frac{M}{n+1} \to 0$ as $n \to \infty$. Hence it follows that $\lim\limits_{n\to\infty} \int\limits_0^1 x^n f(x)\,dx = 0$.

Ex.48 If $f : [0, 2\pi] \to \mathbb{R}$ is continuous such that $\int\limits_0^{\frac{\pi}{2}} f(x)\,dx = 0$, then show that there exists $c \in (0, \frac{\pi}{2})$ such that $f(c) = 2\cos 2c$.
Solution: Let $g(x) = \int\limits_0^x f(t)\,dt - \sin 2x$ for all $x \in [0, 2\pi]$. Since $f : [0, 2\pi] \to \mathbb{R}$ is continuous, by the first fundamental theorem of calculus, $g : [0, 2\pi] \to \mathbb{R}$ is differentiable and $g'(x) = f(x) - 2\cos 2x$ for all $x \in [0, 2\pi]$. Also, $g(0) = 0 = g(\frac{\pi}{2})$ (since $\int\limits_0^{\frac{\pi}{2}} f(x)\,dx = 0$). Hence by Rolle's theorem, there exists $c \in (0, \frac{\pi}{2})$ such that $g'(c) = 0$, *i.e.* $f(c) = 2\cos 2c$.

Ex.49 Prove that for each $a \geq 0$, there exists a unique $b \geq 0$ such that $a = \int_0^b \frac{1}{(1+x^3)^{1/5}}\,dx$.

Solution: Let $a \geq 0$ and let $F(y) = \int_0^y \frac{1}{(1+x^3)^{1/5}}\,dx$ for all $y \geq 0$. Since $\frac{1}{(1+x^3)^{1/5}}$ is continuous for all $x \in [0,\infty)$, by the first fundamental theorem of calculus, $F : [0,\infty) \to \mathbb{R}$ is differentiable and $F'(y) = \frac{1}{(1+y^3)^{1/5}} > 0$ for all $y \in [0,\infty)$. Hence F is strictly increasing on $[0,\infty)$ and so there can be at most one $b \geq 0$ satisfying $F(b) = a$. If $a = 0$, we take $b = 0$. We now assume that $a > 0$. We have $F(y) \geq \int_1^y \frac{1}{(1+x^3)^{1/5}}\,dx \geq \int_1^y \frac{1}{(2x^3)^{1/5}}\,dx = \frac{5}{2^{6/5}}(y^{\frac{2}{5}} - 1) \to \infty$ as $y \to \infty$. Hence there exists $y_1 > 0$ such that $F(0) < a < F(y_1)$. Since F is continuous, by the intermediate value theorem, there exists $b \in (0, y_1)$ such that $F(b) = a$.

Ex.50 Show that there exists a positive real number α such that $\int_0^\pi x^\alpha \sin x\,dx = 3$.

Hint: The function $f : [0,1] \to \mathbb{R}$, defined by $f(\lambda) = \int_0^\pi x^\lambda \sin x\,dx$ for all $\lambda \in [0,1]$, can be shown to be continuous. Also, $f(0) = \int_0^\pi \sin x\,dx = 2 < 3$ and $f(1) = \int_0^\pi x \sin x\,dx = \pi > 3$. Hence by the intermediate value property of continuous functions, there exists $\alpha \in (0,1)$ such that $f(\alpha) = \int_0^\pi x^\alpha \sin x\,dx = 3$.

Ex.51 Determine all real values of p for which the integral $\int_0^\infty \frac{e^{-x}-1}{x^p}\,dx$ is convergent.

Solution: The given integral is convergent iff both $\int_0^1 \frac{1-e^{-x}}{x^p}\,dx$ and $\int_1^\infty \frac{1-e^{-x}}{x^p}\,dx$ are convergent. If $p \leq 0$, then $\int_0^1 \frac{1-e^{-x}}{x^p}\,dx$ exists (in $\mathbb{R}$) as a Riemann integral. For $p > 0$, since $\lim\limits_{x\to 0+}(\frac{1-e^{-x}}{x^p} \cdot x^{p-1}) = \lim\limits_{x\to 0+}(e^{-x} \cdot \frac{e^x-1}{x}) = 1 \neq 0$, by the limit comparison test, $\int_0^1 \frac{1-e^{-x}}{x^p}\,dx$ converges iff $\int_0^1 \frac{1}{x^{p-1}}\,dx$ converges. We know that $\int_0^1 \frac{1}{x^{p-1}}\,dx$ converges iff $p-1 < 1$, *i.e.* iff $p < 2$. Hence $\int_0^1 \frac{1-e^{-x}}{x^p}\,dx$ converges iff $p < 2$. Again, since $\lim\limits_{x\to\infty}(\frac{1-e^{-x}}{x^p} \cdot x^p) = \lim\limits_{x\to\infty}(1 - e^{-x}) = 1 \neq 0$, by the limit comparison test, $\int_1^\infty \frac{1-e^{-x}}{x^p}\,dx$ converges iff $\int_1^\infty \frac{1}{x^p}\,dx$ converges. We know that $\int_1^\infty \frac{1}{x^p}\,dx$ converges iff $p > 1$. Hence $\int_1^\infty \frac{1-e^{-x}}{x^p}\,dx$ converges iff $p > 1$. Therefore the given integral is convergent iff $1 < p < 2$.