# MA 101 (Mathematics I)

## Model Solutions of End-semester Examination (Calculus)

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**4.(a)** Let $\alpha = \lim\limits_{x\to 1+} \frac{f(x)-f(1)}{x-1}$, so that $\alpha \in \mathbb{R}$. Then $\lim\limits_{x\to 1+} [f(x)-f(1)] = \lim\limits_{x\to 1+} \left[\frac{f(x)-f(1)}{x-1} \cdot (x-1)\right] = \lim\limits_{x\to 1+} \frac{f(x)-f(1)}{x-1} \cdot \lim\limits_{x\to 1+} (x-1) = \alpha.0 = 0$ and so $\lim\limits_{x\to 1+} f(x) = \lim\limits_{x\to 1+} [f(x)-f(1)+f(1)] = \lim\limits_{x\to 1+} [f(x)-f(1)] + \lim\limits_{x\to 1+} f(1) = 0 + f(1) = f(1)$. Similarly, we get $\lim\limits_{x\to 1-} f(x) = f(1)$. Consequently $\lim\limits_{x\to 1} f(x) = f(1)$ and so $f$ is continuous at 1. Therefore the given statement is TRUE.

**(b)** For each $n \in \mathbb{N}$, let $x_n = \begin{cases} 0 & \text{if } n \text{ is prime,} \\ 1 & \text{if } n \text{ is not prime.} \end{cases}$
Then for all $m, n \in \mathbb{N} \setminus \{1\}$, $x_{mn} = 1$ and so for each $m \in \mathbb{N} \setminus \{1\}$, $x_{mn} \to 1$. However, since the subsequence $(x_p)$ (with $p$ varying over all primes) of $(x_n)$ converges to 0, the sequence $(x_n)$ cannot be convergent. Therefore the given statement is FALSE.

**(c)** If the power series $\sum\limits_{n=0}^{\infty} a_n(x-3)^n$ is (conditionally) convergent for $x = -5$, then the power series is (absolutely) convergent for all $x \in \mathbb{R}$ satisfying $|x-3| < |-5-3| = 8$ and hence it must be convergent for $x = 8$. Therefore the given statement is FALSE.

**(d)** If $f(x) = \sqrt{(x-1)(2-x)}$ for all $x \in [1,2]$, then $f : [1,2] \to \mathbb{R}$ is continuous and $f$ is differentiable on $(1,2)$. Since $\lim\limits_{x\to 1+} \frac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1+} \frac{\sqrt{2-x}}{\sqrt{x-1}}$ and $\lim\limits_{x\to 2-} \frac{f(x)-f(2)}{x-2} = -\lim\limits_{x\to 2-} \frac{\sqrt{x-1}}{\sqrt{2-x}}$ do not exist (in $\mathbb{R}$), $f$ is not differentiable at 1 and 2. Therefore the given statement is TRUE.

**(e)** Let $f(x) = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{if } 1 < x \le 2. \end{cases}$
Then $f : [1,2] \to \mathbb{R}$ is Riemann integrable on $[1,2]$ and $\int_1^2 f(x)\,dx = 0$, since for every partition $P$ of $[1,2]$, $L(f,P) = 0$, $U(f,P) \ge 0$ and if $0 < \varepsilon < 1$, then for the partition $P = \{1, 1+\frac{\varepsilon}{2}, 2\}$ of $[1,2]$, $U(f,P) = \frac{\varepsilon}{2} < \varepsilon$. Hence it follows that $F(x) = \int_1^x f(t)\,dt = 0$ for all $x \in [1,2]$. Thus $F : [1,2] \to \mathbb{R}$ is differentiable on $[1,2]$ but $F'(1) = 0 \ne f(1)$. Therefore the given statement is FALSE.

**5.** Let $a_n = x_n + 3\left(\frac{n}{n+1}\right)^n$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{2}{3} < 1$ and hence by root test, the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent. Consequently $\lim\limits_{n\to\infty} a_n = 0$. Therefore $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \left[a_n - \frac{3}{(1+\frac{1}{n})^n}\right] = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} \frac{3}{(1+\frac{1}{n})^n} = 0 - \frac{3}{e} = -\frac{3}{e}$.

**6.** Let $x_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^p} = \frac{1}{n^p(\sqrt{n+1}+\sqrt{n})}$ and $y_n = \frac{1}{n^{p+\frac{1}{2}}}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2} \ne 0$. Since the series $\sum\limits_{n=1}^{\infty} y_n$ is convergent iff $p + \frac{1}{2} > 1$, *i.e.* iff $p > \frac{1}{2}$, by the limit comparison test, the series $\sum\limits_{n=1}^{\infty} x_n$ is convergent iff $p > \frac{1}{2}$.

**7.** Let $|f(a)| = \min\{|f(-1)|, |f(0)|, |f(1)|\}$ and $|f(b)| = \max\{|f(-1)|, |f(0)|, |f(1)|\}$, where $a, b \in \{-1, 0, 1\}$. Then $|f|(a) = |f(a)| \le \frac{1}{4}\big(|f(-1)| + 2|f(0)| + |f(1)|\big) \le |f(b)| = |f|(b)$.

Since $f$ is continuous, the function $|f| : [-1.1] \to \mathbb{R}$ is also continuous. Hence by the intermediate value property of the continuous function $|f|$, there exists $c \in [-1,1]$ such that $|f(c)| = |f|(c) = \frac{1}{4}\big(|f(-1)| + |f(0)| + |f(1)|\big)$.

**8.** Since $f$ is differentiable on $[0,1]$, $f$ is continuous on $[0,1]$. Since $f(0) < \frac{1}{2} < f(1)$, by the intermediate value property of the continuous function $f$, there exists $c \in (0,1)$ such that $f(c) = \frac{1}{2}$. Applying the mean value theorem on $[0,c]$ and $[c,1]$, there exist $a \in (0,c)$ and $b \in (c,1)$ such that $f(c) - f(0) = cf'(a)$ and $f(1) - f(c) = (1-c)f'(b)$. Thus $a \neq b$, $f'(a) = \frac{1}{2c}$ and $f'(b) = \frac{1}{2(1-c)}$. Hence $\frac{1}{f'(a)} + \frac{1}{f'(b)} = 2c + 2(1-c) = 2$.

**9.** If $f(x) = \log(1+x)$ for all $x \in (-\frac{1}{2}, 1)$, then $f : (-\frac{1}{2}, 1) \to \mathbb{R}$ is infinitely differentiable and $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ for all $x \in (-\frac{1}{2}, 1)$ and for all $n \in \mathbb{N}$. Let $x \in (-\frac{1}{2}, 1)$. The remainder term in the Taylor expansion of $f(x)$ about the point 0 is given by $R_n(x) = \frac{f^{n+1}(c_n)}{(n+1)!}x^{n+1} = \frac{(-1)^n x^{n+1}}{(n+1)(1+c_n)^{n+1}}$, where $c_n$ lies between 0 and $x$. If $x \geq 0$, then $1 + c_n > 1$ and so $|R_n(x)| \leq \frac{1}{n+1} \to 0$ as $n \to \infty$. On the other hand, if $x < 0$, then since $1 + c_n > 1 + x > 0$, we get $\frac{1}{1+c_n} < \frac{1}{1+x}$ and so $|R_n(x)| \leq \frac{1}{n+1}\left(\frac{|x|}{1+x}\right)^{n+1} \leq \frac{1}{n+1} \to 0$ as $n \to \infty$ (since $x > -\frac{1}{2}$, we get $\frac{|x|}{1+x} = -\frac{x}{1+x} \leq 1$). Hence for each $x \in (-\frac{1}{2}, 1)$, we have found that $\lim\limits_{n\to\infty} R_n(x) = 0$. Therefore the Taylor series of $\log(1+x)$ about 0 converges to $\log(1+x)$ for each $x \in (-\frac{1}{2}, 1)$.

**10.** Let $f(x) = x^{1+\frac{1}{x}}$ for all $x > 0$. Taking logarithm and differentiating, we get $f'(x) = x^{1+\frac{1}{x}}[\frac{1}{x}(1+\frac{1}{x}) - \frac{1}{x^2}\log x] = x^{\frac{1}{x}}(1 + \frac{1}{x} - \frac{\log x}{x})$ for all $x > 0$. Using L'Hôpital's rule, we find that $\lim\limits_{x\to\infty} \frac{\log x}{x} = 0$ and $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = 1$ (after taking logarithm). Hence $\lim\limits_{x\to\infty} f'(x) = 1$. Therefore $\lim\limits_{x\to\infty}[(x+1)^{\frac{x+2}{x+1}} - x^{\frac{x+1}{x}}] = \lim\limits_{x\to\infty}[f(x+1) - f(x)] = \lim\limits_{x\to\infty} f'(c_x) = 1$, since by the mean value theorem, for each $x > 0$, there exists $c_x \in (x, x+1)$ such that $f(x+1) - f(x) = f'(c_x)$ and since $x < c_x < x+1 \Rightarrow \lim\limits_{x\to\infty} c_x = \infty$.

**11.** The given integral is convergent iff both $\int_1^2 \frac{\sqrt{x+3}}{(x+2)\sqrt{x^2-1}}\,dx$ and $\int_2^\infty \frac{\sqrt{x+3}}{(x+2)\sqrt{x^2-1}}\,dx$ are convergent. Let $f(x) = \frac{\sqrt{x+3}}{(x+2)\sqrt{x^2-1}}$, $g(x) = \frac{1}{\sqrt{x-1}}$ and $h(x) = \frac{1}{x^{\frac{3}{2}}}$ for all $x > 1$. Then $\lim\limits_{x\to1+} \frac{f(x)}{g(x)} = \lim\limits_{x\to1+} \frac{\sqrt{x+3}}{(x+2)\sqrt{x+1}} = \frac{\sqrt{2}}{3}$ and $\lim\limits_{x\to\infty} \frac{f(x)}{h(x)} = \lim\limits_{x\to\infty} \frac{\sqrt{1+\frac{3}{x}}}{(1+\frac{2}{x})\sqrt{1-\frac{1}{x^2}}} = 1$. Since $\int_1^2 g(x)\,dx$ and $\int_2^\infty h(x)\,dx$ are convergent, by the limit comparison test, $\int_1^2 f(x)\,dx$ and $\int_2^\infty f(x)\,dx$ are convergent. Therefore the given integral is convergent.

**12.** The given circle and the cardioid meet at two points corresponding to $\theta = \frac{\pi}{2}$ and $\theta = \pi$. The required area is $\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(3\sin\theta)^2\,d\theta - \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} 9(1+\cos\theta)^2\,d\theta = -\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(9 + 9\cos 2\theta + 18\cos\theta)\,d\theta = 9(1 - \frac{\pi}{4})$.

**13.** The sides of the triangle lie on the lines $y = 2x - 1$, $y = -x + 5$ and $y = \frac{1}{2}(x+1)$. Therefore the required volume is $\pi\int_1^2[(2x-1)^2 - \frac{1}{4}(x+1)^2]\,dx + \pi\int_2^3[(-x+5)^2 - \frac{1}{4}(x+1)^2]\,dx = 6\pi$.