MA 101 (Mathematics I)

Differentiation : Summary of Lectures

Differentiability and Derivative: Let $D \subseteq \mathbb{R}$ and let $x_0 \in D$ such that there exists an interval I of $\mathbb{R}$ satisfying $x_0 \in I \subseteq D$.
A function $f : D \to \mathbb{R}$ is said to be differentiable at x_0 if $\lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ (or equivalently, $\lim\limits_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$) exists in $\mathbb{R}$.
If f is differentiable at x_0, then the derivative of f at x_0 is $f'(x_0) = \lim\limits_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim\limits_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$.
$f : D \to \mathbb{R}$ is said to be differentiable if f is differentiable at each $x_0 \in D$.

Result: If $f : D \to \mathbb{R}$ is differentiable at $x_0 \in D$, then f is continuous at x_0.

Examples:

1. For $n = 1, 2, 3$, let $f_n(x) = \begin{cases} x^n \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
2. $f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Rules for finding derivatives:

Definition: $f : D \to \mathbb{R}$ has a local maximum (resp. minimum) at $x_0 \in D$ if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ (resp. $f(x_0) \leq f(x)$) for all $x \in (x_0 - \delta, x_0 + \delta) \cap D$.

Result: If $f : D \to \mathbb{R}$ has a local maximum or local minimum at an interior point x_0 of D and if f is differentiable at x_0, then $f'(x_0) = 0$.

Rolle's theorem: If $f : [a, b] \to \mathbb{R}$ is continuous, if f is differentiable on (a, b) and if $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Examples:
(a) The equation $x^2 = x \sin x + \cos x$ has exactly two real roots.
(b) The equation $x^4 + 2x^2 - 6x + 2 = 0$ has exactly two real roots.

Mean value theorem: If $f : [a, b] \to \mathbb{R}$ is continuous and if f is differentiable on (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Result: Let $f : I \to \mathbb{R}$ be differentiable. Then
(a) $f'(x) = 0$ for all $x \in I$ iff f is constant on I.
(b) $f'(x) \geq 0$ for all $x \in I$ iff f is increasing on I.
(c) $f'(x) \leq 0$ for all $x \in I$ iff f is decreasing on I.
(d) $f'(x) > 0$ for all $x \in I \Rightarrow f$ is strictly increasing on I.
(e) $f'(x) < 0$ for all $x \in I \Rightarrow f$ is strictly decreasing on I.
(f) $f'(x) \neq 0$ for all $x \in I \Rightarrow f$ is one-one on I.

Examples:
(a) $\sin x \geq x - \frac{x^3}{6}$ for all $x \in [0, \frac{\pi}{2}]$.

(b) If $f(x) = x^3 + x^2 - 5x + 3$ for all $x \in \mathbb{R}$, then f is one-one on $[1, 5]$ but not one-one on $\mathbb{R}$.

Intermediate value property of derivatives: Let $f : I \to \mathbb{R}$ be differentiable and let $a, b \in I$ with $a < b$. If $f'(a) < k < f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = k$.

Example: Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(-1) = 5$, $f(0) = 0$ and $f(1) = 10$. Then there exist c_1, $c_2 \in (-1, 1)$ such that $f'(c_1) = -3$ and $f'(c_2) = 3$.

Local maximum & Local minimum : Sufficient conditions:

1. First derivative test
2. Second derivative test

Example: Local maxima and local minima of f, where $f(x) = 1 - x^{2/3}$ for all $x \in \mathbb{R}$.

L'Hôpital's rules:

1. Let $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ be differentiable at $x_0 \in (a, b)$. Also, let $f(x_0) = g(x_0) = 0$ and $g'(x_0) \neq 0$. Then $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$.
2. Let $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ be differentiable such that $\lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a+} g(x) = 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim\limits_{x \to a+} \frac{f'(x)}{g'(x)} = \ell$, then $\lim\limits_{x \to a+} \frac{f(x)}{g(x)} = \ell$.

Examples: (a) $\lim\limits_{x \to 0} \frac{\sqrt{1+x}-1}{x}$ (b) $\lim\limits_{x \to \frac{\pi}{2}} \frac{1-\sin x}{1+\cos 2x}$ (c) $\lim\limits_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$ (d) $\lim\limits_{x \to 0} (\frac{\sin x}{x})^{\frac{1}{x}}$
(e) $\lim\limits_{x \to \infty} \frac{x - \sin x}{2x + \sin x}$

Taylor's theorem: Let $f : [a, b] \to \mathbb{R}$ be such that $f, f', f'', ..., f^{(n)}$ are continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b). Then there exists $c \in (a, b)$ such that
$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b - a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b - a)^{n+1}$.

Example: $1 + \frac{x}{2} - \frac{x^2}{8} \leq \sqrt{1 + x} \leq 1 + \frac{x}{2}$ for all $x > 0$.

Power series: A series of the form $\sum\limits_{n=0}^{\infty} a_n(x - x_0)^n$,
where x_0, $a_n \in \mathbb{R}$ for $n = 0, 1, 2, ...$ and $x \in \mathbb{R}$.
It is sufficient to consider the series $\sum\limits_{n=0}^{\infty} a_n x^n$.

Convergence - Examples:
(a) $\sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ (b) $\sum\limits_{n=0}^{\infty} n! x^n$ (c) $\sum\limits_{n=0}^{\infty} x^n$

Result:

(a) If $\sum\limits_{n=0}^{\infty} a_n x^n$ converges for $x = x_1 \neq 0$, then it converges absolutely for all $x \in \mathbb{R}$ satisfying $|x| < |x_1|$.

(b) If $\sum\limits_{n=0}^{\infty} a_n x^n$ diverges for $x = x_2$, then it diverges for all $x \in \mathbb{R}$ satisfying $|x| > |x_2|$.

Radius of convergence: For every power series $\sum\limits_{n=0}^{\infty} a_n x^n$, there exists a unique R satisfying $0 \leq R \leq \infty$ such that the series converges absolutely if $|x| < R$ and diverges if $|x| > R$.

The series may or may not converge for $|x| = R$.

Methods to find the radius/interval of convergence:

Examples: (a) $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ (b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 4^n}(x-1)^n$

Term-by-term operations on power series:

Taylor series & Maclaurin series: Convergence

Examples: Taylor series expansions of e^x, $\sin x$ and $\cos x$.

Result on local maxima and local minima:

Let $x_0 \in (a, b)$ and let $n \geq 2$. Also, let $f, f', ..., f^{(n)}$ be continuous on (a, b) and $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ but $f^{(n)}(x_0) \neq 0$.

(a) If n is even and $f^{(n)}(x_0) < 0$, then f has a local maximum at x_0.
(b) If n is even and $f^{(n)}(x_0) > 0$, then f has a local minimum at x_0.
(c) If n is odd, then f has neither a local maximum nor a local minimum at x_0.

Example: Local maximum and local minimum values of f, where
$f(x) = x^5 - 5x^4 + 5x^3 + 12$ for all $x \in \mathbb{R}$.