Problem Set 2

Turbulent Flows (ME 695)

Department of Mechanical Engineering

Indian Institute of Technology Guwahati

1. Turbulent-viscosity hypothesis relates deviatoric Reynolds stress with the mean strain rate given by

$$-\rho\langle u_i'u_j'\rangle + \frac{2}{3}\rho k\delta_{ij} = \rho\nu_T(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i})$$

where ν_T is turbulent viscosity, U_i, u_i' are mean and fluctuating velocities and $k = 1/2\langle u_i'u_i'\rangle$ is the turbulent kinetic energy with $\langle . \rangle$ means averaging. Show that in order to yield non-negative normal stresses, it is necessary and sufficient for the turbulent viscosity to satisfy

$$\nu_T \leq \frac{k}{3S_\lambda}$$

with S_λ being the largest eigenvalue of the mean strain-rate tensor.

2. Dynamical equation for the mean squared vorticity fluctuation $1/2\langle\omega_i'\omega_i'\rangle$ can be derived by a procedure identical to the one followed for the turbulent kinetic energy. Derive it and explain significance of each term. Note $\omega_i = \Omega_i + \omega_i'$.

$$\begin{aligned}\frac{D}{Dt}(1/2\langle\omega_i'\omega_i'\rangle) = \quad & -\langle u_j'\omega_i'\rangle\frac{\partial\Omega_i}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial x_j}\langle u_j'\omega_i'\omega_i'\rangle + \langle\omega_i'\omega_j's_{ij}'\rangle + \langle\omega_i'\omega_j'\rangle S_{ij} \\ & +\Omega_j\langle\omega_i's_{ij}'\rangle + \nu\frac{\partial^2}{\partial x_j\partial x_j}(1/2\langle\omega_i'\omega_i'\rangle) - \nu\langle\frac{\partial\omega_i'}{\partial x_j}\frac{\partial\omega_i'}{\partial x_j}\rangle\end{aligned}$$

3. If pseudo-dissipation of turbulent kinetic energy is defined as $\tilde{\epsilon} = \nu\langle\partial u_i'/\partial x_j \partial u_i'/\partial x_j\rangle$ then show that the alternative form of the turbulent kinetic energy equation is

$$(\frac{\partial}{\partial t} + U_j\frac{\partial}{\partial x_j})k = -\frac{\partial}{\partial x_j}[1/2\langle u_i'u_i'u_j'\rangle + \langle u_j'p'\rangle/\rho] + \nu\nabla^2 k + P - \tilde{\epsilon}$$

where P is the production of turbulent kinetic energy. Also find the difference between $\tilde{\epsilon}$ and true dissipation.

4. Starting from the Fourier representation for the velocity prove that

$$\left\langle\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_l}\right\rangle = \sum_{\boldsymbol{\kappa}}\kappa_k\kappa_l\hat{R}_{ij}(\boldsymbol{\kappa}, t) = \iiint_{-\infty}^{\infty}\overline{\kappa}_k\overline{\kappa}_l\Phi_{ij}(\overline{\boldsymbol{\kappa}}, t)d\overline{\boldsymbol{\kappa}}$$

and hence show that the dissipation rate $\epsilon(t)$ is given by

$$\epsilon(t) = \sum_{\boldsymbol{\kappa}} 2\nu\kappa^2\hat{E}(\boldsymbol{\kappa}, t) = \iiint_{-\infty}^{\infty} 2\nu\overline{\kappa}^2(1/2)\Phi_{ii}(\overline{\boldsymbol{\kappa}}, t)d\overline{\boldsymbol{\kappa}}$$

where $\Phi_{ij}(\boldsymbol{\kappa}, t)$ is the velocity spectrum tensor.

5. In isotropic turbulence directional information of the velocity spectrum tensor $\Phi_{ij}(\boldsymbol{\kappa}, t)$ can depend only on $\boldsymbol{\kappa}$. Since the only second-order isotropic tensors that can be formed from $\boldsymbol{\kappa}$ are δ_{ij} and $\kappa_i\kappa_j$ show that

energy spectrum function $E(\kappa, t)$ can completely describe $\Phi_{ij}(\boldsymbol{\kappa}, t)$ by

$$\Phi_{ij}(\boldsymbol{\kappa}, t) = \frac{E(\kappa, t)}{4\pi\kappa^2} P_{ij}(\boldsymbol{\kappa})$$

where $P_{ij}(\boldsymbol{\kappa})$ is the matrix that projects a vector onto the plane normal to $\boldsymbol{\kappa}$.

6. Show the evolution equation for kinetic energy of the Fourier mode, defined as $\hat{E}(\boldsymbol{\kappa}, t) = (1/2)\langle \hat{u}_i^* \hat{u}_i \rangle$, is given by

$$\left(\frac{\partial}{\partial t} + 2\nu\kappa^2 \right) \hat{E}(\boldsymbol{\kappa}, t) = \hat{T}(\boldsymbol{\kappa}, t) = \kappa_j P_{il} \sum_{\boldsymbol{\kappa}'} \text{Real} \left(\hat{i} \langle \hat{u}_l^*(\boldsymbol{\kappa}', t) \hat{u}_j^*(\boldsymbol{\kappa} - \boldsymbol{\kappa}', t) \hat{u}_i(\boldsymbol{\kappa}, t) \rangle \right)$$

where $\hat{i} = \sqrt{-1}$ and ϕ^* is the complex conjugate of ϕ. By integrating the above equation over all $\boldsymbol{\kappa}$ obtain the kinetic energy equation in the physical space

$$\frac{dk}{dt} = -\epsilon$$

and explain the nature and role of the term $\hat{T}(\boldsymbol{\kappa}, t)$.

7. Using the definition of the energy spectrum function $E(\kappa) = \oint (1/2)\Phi_{ii}(\boldsymbol{\kappa}) dS_\kappa$ and one-dimensional spectra $E_{ij}(\kappa_1) = (1/\pi) \int_{-\infty}^{\infty} R_{ij}(r_1) e^{-\hat{i}\kappa_1 r_1} dr_1$ show that the one-dimensional spectra can be computed from the energy spectrum function by

$$E_{11}(\kappa_1) = \int_{\kappa_1}^{\infty} \frac{E(\kappa)}{\kappa} \left(1 - \frac{\kappa_1^2}{\kappa^2} \right) d\kappa$$

Also show that $E_{11}(\kappa_1)$ is a monotonically decreasing function of κ_1 with maximum at $\kappa_1 = 0$. Here $\oint ... dS_\kappa$ implies integral over the surface of the sphere with radius κ.

8. Consider the model energy-spectrum function

$$E(\kappa) = C\epsilon^{2/3}\kappa^{-5/3} f_L(\kappa L) f_\eta(\kappa\eta)$$

where f_L and f_η are non-dimensional functions that describe the shape of $E(\kappa)$ in the energy containing and dissipation ranges, respectively. Find out the properties of these functions so that the Kolmogorov -5/3 law is recovered. Show that the expression for dissipation obtained from integration of the model spectrum is

$$\epsilon = 2C\nu\epsilon^{2/3}\eta^{-4/3} \int_0^{\infty} (\kappa\eta)^{1/3} f_\eta(\kappa\eta) d(\kappa\eta)$$

Now show that if the dissipation part of the spectrum is modeled as $f_\eta(\kappa\eta) = \exp(-\beta_0\kappa\eta)$ then the above integral is

$$\int_0^{\infty} x^{1/3} \exp(-\beta_0 x) dx = \beta_0^{-4/3} \Gamma(4/3)$$

Hence show that in the limit of high Reynolds number $\beta_0 \approx 2.094$. Note $\Gamma(x) = \int_0^{\infty} \exp(-t) t^{x-1} dt$ is the Gamma function, L, η are length of largest and smallest scale eddies.