Problem Set 1

Turbulent Flows (ME 695)

Department of Mechanical Engineering

Indian Institute of Technology Guwahati

1. Use Reynolds transport theorem to deduce the basic form of the momentum equation with respect to volume and surface forces. Explain how surface force on a fluid particle can be expressed as $n_j\sigma_{ij}$ where n_i and σ_{ij} are the orientation of the surface and stress tensor. By using the idea of isotropic physical processes construct the stress tensor for Newtonian fluids. Finally derive the well known form of the momentum equation for incompressible flows - *Navier-Stokes equations*. Derive the mass balance equation if required at any step.

2. If e_r and e_θ are the unit vectors in r and θ directions, respectively, then deduce $\partial e_r/\partial\theta$ and $\partial e_\theta/\partial\theta$. Expand the incompressible Navier-Stokes equations for both Cartesian and cylindrical coordinate systems. Also find out the components of the shear stress tensor in both the coordinate systems.

3. Angular momentum of a fluid particle (having velocity u_i) per unit volume is defined as $\boldsymbol{M} = \rho\boldsymbol{r}\times\boldsymbol{u}$. By forming a cross product of the momentum equation with $\boldsymbol{r}$ show that the moment of momentum equation is

$$(\partial_t + \boldsymbol{u}.\nabla)\boldsymbol{M} = \rho\boldsymbol{r}\times\boldsymbol{f} + \boldsymbol{r}\times(\nabla.\overline{\overline{\sigma}})$$

where $\boldsymbol{f}$ and $\overline{\overline{\sigma}}$ are volumetric forces per unit mass and stress tensor, respectively. Now, if internal moment per unit mass due to molecular motion be $\boldsymbol{m}$ and surface couple due to molecular transport through an imaginary surface be $\boldsymbol{n}.\overline{\overline{\Omega}}$ show that the conservation law for net angular momentum $\rho\boldsymbol{r}\times\boldsymbol{u}+\boldsymbol{m}$ is

$$(\partial_t + \boldsymbol{u}.\nabla)(\boldsymbol{M}+\rho\boldsymbol{m}) = \rho\boldsymbol{r}\times\boldsymbol{f} + \nabla.(\boldsymbol{r}\times\overline{\overline{\sigma}}) + \nabla.\overline{\overline{\Omega}}$$

Derive the conservation law for internal angular momentum $\boldsymbol{m}$ by subtracting the moment of momentum equation from the net angular momentum equation

$$(\partial_t + u_j\partial_j)(\rho m_i) = \epsilon_{ijk}\sigma_{jk} + \partial_j\Omega_{ji}$$

Finally show that the stress tensor is symmetric if internal angular momentum and surface couple is zero owing to random motion of the molecules.

4. Write down the equation for the stream-wise (x) velocity (u) for fully developed flow in a channel with elliptical cross-section. Consider a length scale L and by taking equality of pressure gradient and viscous force find out the velocity scale, and using them non-dimensionalize the above equation. Introduce a transformation $U = u + a_1y^2 + a_2z^2$ (where y, z are coordinates scaled by L) and manipulate the boundary condition for U to find the Laplace equation for U. Now use extremum principle of the Laplace equation to solve for u.

5. A double film of thickness H consisting of two immiscible liquids is falling along an smooth inclined plane that makes an angle α with the horizontal direction. If the height of the interface is h find the velocity distribution in the film.

6. Consider the steady flat plate boundary layer with outer inviscid velocity $U(x)$

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \\
\frac{\partial p}{\partial y} &= 0
\end{aligned}$$

Find the nature of variation of $U(x)$ so that the boundary layer equations written above admit similarity solution.

7. Using the result from Prob. 6 attempt for similarity solution for a flat plate boundary layer with a sink at the leading edge where the outer flow is governed by $U(x) = m/x, m < 0$. Use velocity scale as $U(x)$ and length scale $\delta(x)$. Transform the boundary layer equation in terms of stream function using $u = -\partial\psi/\partial y,\ v = \partial\psi/\partial x$, introduce the similarity function $f(\eta)$ in the scaling of ψ and finally obtain the ordinary differential equation $f''' - f'^2 = -1$. Also obtain the boundary conditions for f.

8. Starting from the equation of motion

$$\rho\frac{Du_i}{Dt} = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j}$$

and contracting it by u_i show that the kinetic energy equation can be written as

$$\rho\frac{D}{Dt}(\frac{1}{2}u_i u_i) = \rho\boldsymbol{g}\cdot\boldsymbol{u} + \frac{\partial}{\partial x_j}(u_i\sigma_{ij}) + p(\nabla\cdot\boldsymbol{u}) - \phi$$

where $\boldsymbol{g}, \boldsymbol{u}, \sigma_{ij}, \phi$ are body force per unit mass, velocity, stress tensor and viscous dissipation, respectively. Explain the meaning and importance of each term.

9. For entry flow between two parallel plates with vertical separation $2h$ show that the entrance length scales as $(L_e/h) \sim Re$. Now using the idea of displacement thickness (δ^*), and the approximate solution $\delta(x)/x = 5Re_x^{-1/2},\ \delta^* = 0.344\delta$ show that $Le/h \approx 0.03Re$ where $Re = 2U_0 h/\nu$, U_0 inlet uniform velocity.

10. The continuity and momentum equations in a coordinate frame rotating with a constant angular velocity $\boldsymbol{\Omega}$ is given by

$$\begin{aligned}
\nabla\cdot\boldsymbol{u} &= 0 \\
\frac{D\boldsymbol{u}}{Dt} &= -\frac{1}{\rho}\nabla p - 2\boldsymbol{\Omega}\times\boldsymbol{u} + \nu\nabla^2\boldsymbol{u}
\end{aligned}$$

Show that for constant density flows by casting the above equation appropriately and subsequently taking curl of them produces the vorticity ($\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$) equation in the rotating frame reference

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} + 2\boldsymbol{\Omega})\cdot\nabla\boldsymbol{u} + \nu\nabla^2\boldsymbol{\omega}$$