Problem Set 1

Advanced Engineering Mathematics (ME 501)
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1. For what right-hand sides (b) the system $(Ax = b)$ is solvable when

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 4 & 2 & 3 \\ 2 & -8 & 1 & -1 \\ -2 & 4 & 3 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 1 \end{bmatrix}$$

2. Find the rank of the following matrices

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 8 \\ -3 & 4 & 4 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 & 5 \\ 2 & -4 & 6 \\ 10 & 0 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

3. Find the Row space, Column space and the Null space of the following matrices

$$\begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 9 \\ -1 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix}$$

4. Prove that for a matrix $A \in \boldsymbol{R^{m \times n}}$ number of linearly independent rows and columns are identically same.

5. Is it possible to find a set of n linearly independent vectors in $\boldsymbol{R^m}$ where $n > m$?

6. Consider a linear system $Ax = b$ with $A \in \boldsymbol{R^{m \times n}},\ x \in \boldsymbol{R^n}, b \in \boldsymbol{R^m}$. If **rank** of the matrix A is **as large as possible** then prove that
(a) Columns span R^m and the system has **at least** one solution for $m < n$.
(b) The system has **at most** one solution for $m > n$.

7. Find the dimension of the subspace of R^4 spanned by the set $(1, 0, 0, 0)$,$(0, 1, 0, 0)$,$(1, 2, 0, 1)$, $(0, 0, 0, 1)$. Hence find its basis.

8. Under what conditions on their entries are A and B invertible?

$$\begin{bmatrix} a & b & c \\ d & e & 0 \\ f & 0 & 0 \end{bmatrix}, \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}, \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

9. Find the complete solutions of

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned} \qquad \begin{aligned} x + 3y + z + 2t &= 1 \\ 2x + 6y + 4z + 8t &= 3 \\ 2z + 4t &= 1 \end{aligned} \qquad \begin{aligned} x + 2z + 3t &= 4 \\ x + 3y + 2z &= 5 \\ 2x + 4z + 9t &= 10 \end{aligned} \qquad \begin{aligned} x - 3y + 2t &= 3 \\ 2x + 5y + 3z + 7t &= -4 \\ 2z + 4t &= 1 \\ x - 3y + 2z + 6t &= 4 \end{aligned}$$

10. Find the dimension and construct a basis for the Row, Column, Null and left Null spaces for

the matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

11. Find the matrix $\boldsymbol{A}$ which indicates the following linear transformations. Find its eigenvalues and eigenvectors and explain their significance.
(a) Reflection about the x-axis in R^2.
(b) Counterclockwise rotation through the angle $\pi/2$ about the origin in R^2.
(c) Reflection about the xy-plane in R^3.
(d) Projection of R^2 onto the y-axis.
(e) Projection of R^3 onto the plane $y = x$.

12. Prove the following
(a) Main diagonal entries of a skew-symmetric matrix must be zero.
(b) Eigenvectors corresponding to distinct eigenvalues are orthogonal for a symmetric matrix.
(c) Inverse of a skew-symmetric matrix is skew-symmetric.
(d) If A has distinct eigenvalues and $AB = BA$ for a matrix B, then matrices A and B share eigenvectors.
(e) Eigenvectors corresponding to distinct eigenvalues of a unitary matrix are orthonormal.
(f) Eigenvalues of A and A^T are same. Show by an example that eigenvectors of A and A^T are not the same.

13. Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$$

and making a suitable (clever) choice of λ. Next find the coefficient of $(-\lambda)^{n-1}$ on the right side of the above equation. Now find the coefficient of $(-\lambda)^{n-1}$ in

$$\det(A - \lambda I) = \det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

and compare. That proves trace equals the sum of the eigenvalues.

14. Apply the Gram-Schmidt process to the columns of A given by $a_1 = (2, 2, 0)$, $a_2 = (1/2, 0, 1/2)$, $a_3 = (0, 3, 3)$ to find out an orthonormal set q_1, q_2, q_3. If $Q = [q_1 \;\; q_2 \;\; q_3]$, then find R such that $A = QR$. What did you learn from this exercise?

15. Find the eigenvalues and eigenvectors of

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

16. Factor the following matrices into $S\Lambda S^{-1}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$