1. For what right-hand sides \((b)\) the system \((Ax = b)\) is solvable when
\[
A = \begin{bmatrix}
1 & 4 & 2 \\
2 & 8 & 4 \\
-1 & -4 & -2
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
1 & 4 \\
2 & 9 \\
-1 & -4
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
-1 & 4 & 2 & 3 \\
2 & -8 & 1 & -1 \\
-2 & 4 & 3 & 6
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
1 & 1 & 2 \\
2 & 3 & -1
\end{bmatrix}
\]

2. Find the rank of the following matrices
\[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
3 & 1 & 4 \\
0 & 5 & 8 \\
-3 & 4 & 4 \\
1 & 2 & 4
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
3 & -1 & 5 \\
2 & -4 & 6 \\
10 & 0 & 14 \\
1 & 2 & 0 & 1
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{bmatrix}
\]

3. Find the Row space, Column space and the Null space of the following matrices
\[
\begin{bmatrix}
2 & 4 & 6 & 4 \\
2 & 5 & 7 & 6 \\
2 & 3 & 5 & 2
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
1 & 3 & 3 \\
2 & 6 & 9 \\
-1 & -3 & 3
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
1 & 3 & 1 & 2 \\
2 & 6 & 4 & 8 \\
0 & 0 & 2 & 4
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6 \\
2 & 5 & 7 \\
3 & 9 & 12
\end{bmatrix}
\]

4. Prove that for a matrix \(A \in \mathbb{R}^{m \times n}\) number of linearly independent rows and columns are identically same.

5. Is it possible to find a set of \(n\) linearly independent vectors in \(\mathbb{R}^{m}\) where \(n > m\)?

6. Consider a linear system \(Ax = b\) with \(A \in \mathbb{R}^{m \times n}, x \in \mathbb{R}^{n}, b \in \mathbb{R}^{m}\). If rank of the matrix \(A\) is as large as possible then prove that
(a) Columns span \(\mathbb{R}^{m}\) and the system has at least one solution for \(m < n\).
(b) The system has at most one solution for \(m > n\).

7. Find the dimension of the subspace of \(\mathbb{R}^{4}\) spanned by the set \((1, 0, 0, 0),(0, 1, 0, 0),(1, 2, 0, 1), (0, 0, 0, 1)\). Hence find its basis.

8. Under what conditions on their entries are \(A\) and \(B\) invertible?
\[
\begin{bmatrix}
a & b & c \\
d & e & 0 \\
f & 0 & 0
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
a & b & 0 \\
c & d & 0 \\
0 & 0 & e
\end{bmatrix}
\]
\[
, \quad \begin{bmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{bmatrix}
\]

9. Find the complete solutions of
\[
x + 3y + 3z = 1 \quad x + 3y + z + 2t = 1 \quad x + 2z + 3t = 4 \quad x - 3y + 2t = 3
\]
\[
2x + 6y + 9z = 5 \quad 2x + 6y + 4z + 8t = 3 \quad x + 3y + 2z = 5 \quad 2x + 5y + 3z + 7t = -4
\]
\[
-x - 3y + 3z = 5 \quad 2z + 4t = 1 \quad 2x + 4z + 9t = 10 \quad 2z + 4t = 1
\]
\[
x - 3y + 2z + 6t = 4
\]

10. Find the dimension and construct a basis for the Row, Column, Null and left Null spaces for
the matrices
\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 2 & 0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 1 & 4 & 0 \\
0 & 2 & 8 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 4 & 6 \\
0 & 0 & 0 & 1 & 2
\end{bmatrix}
\]

11. Find the matrix \( A \) which indicates the following linear transformations. Find its eigenvalues and eigenvectors and explain their significance.
(a) Reflection about the \( x \)-axis in \( R^2 \).
(b) Counterclockwise rotation through the angle \( \pi/2 \) about the origin in \( R^2 \).
(c) Reflection about the \( xy \)-plane in \( R^3 \).
(d) Projection of \( R^2 \) onto the \( y \)-axis.
(e) Projection of \( R^3 \) onto the plane \( y = x \).

12. Prove the following
(a) Main diagonal entries of a skew-symmetric matrix must be zero.
(b) Eigenvectors corresponding to distinct eigenvalues are orthogonal for a symmetric matrix.
(c) Inverse of a skew-symmetric matrix is skew-symmetric.
(d) If \( A \) has distinct eigenvalues and \( AB = BA \) for a matrix \( B \), then matrices \( A \) and \( B \) share eigenvectors.
(e) Eigenvectors corresponding to distinct eigenvalues of a unitary matrix are orthonormal.
(f) Eigenvalues of \( A \) and \( A^T \) are same. Show by an example that eigenvectors of \( A \) and \( A^T \) are not the same.

13. Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into
\[
\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)
\]
and making a suitable (clever) choice of \( \lambda \). Next find the coefficient of \( (-\lambda)^{n-1} \) on the right side of the above equation. Now find the coefficient of \( (-\lambda)^{n-1} \) in
\[
\det(A - \lambda I) = \det\begin{bmatrix}
a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda
\end{bmatrix}
\]
and compare. That proves trace equals the sum of the eigenvalues.

14. Apply the Gram-Schmidt process to the columns of \( A \) given by \( a_1 = (2, 2, 0), a_2 = (1/2, 0, 1/2), a_3 = (0, 3, 3) \) to find out an orthonormal set \( q_1, q_2, q_3 \). If \( Q = [q_1 \ q_2 \ q_3] \), then find \( R \) such that \( A = QR \). What did you learn from this exercise?

15. Find the eigenvalues and eigenvectors of
\[
\begin{bmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 2 \\
0 & 2 & 0 \\
3 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2
\end{bmatrix}
\]

16. Factor the following matrices into \( SAS^{-1} \)
\[
\begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}, \begin{bmatrix}
2 & 1 \\
0 & 0
\end{bmatrix}
\]