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IWASAWA λ-INVARIANTS AND Γ-TRANSFORMS

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Abstract. In this paper we study a relation between the λ-invariants of a p-adic measure and its Γ-transform exploiting certain combinatorial identities. Along the way we also determine p-adic properties of certain Mahler coefficients.

Key Words: p-adic measure, Γ-transform, Iwasawa invariants, Mahler coefficients.
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1. Introduction

Fix an odd prime p. Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$ with a local parameter π. We write $\mathbb{Z}_p^\times = V \times U$ where V is the group of $(p-1)$st roots of unity in $\mathbb{Z}_p$ and $U = 1 + p\mathbb{Z}_p$. Let u be a topological generator of U. The projections from $\mathbb{Z}_p^\times$ onto V and U are denoted by ω and $<>$ respectively. We have an isomorphism $\phi : \mathbb{Z}_p \to U$ given by $\phi(y) = u^y$.

Let Λ denote the $\mathcal{O}$-valued measures on $\mathbb{Z}_p$. It is well-known, (see e.g. [1]), that Λ is a ring under convolution, and is isomorphic to the formal power series ring $\mathcal{O}[[T-1]]$. Explicitly, for $x \in \mathbb{Z}_p$, let

$$T^x = \sum_{n=0}^{\infty} \binom{x}{n} (T-1)^n \in \mathcal{O}[[T-1]].$$

The power series associated to a measure $\alpha \in \Lambda$ is then defined by

$$\hat{\alpha}(T) = \int_{\mathbb{Z}_p} T^x d\alpha(x) = \sum_{n=0}^{\infty} b_n(\alpha)(T-1)^n$$

where

$$b_n(\alpha) = \int_{\mathbb{Z}_p} \binom{x}{n} d\alpha(x).$$

A classical theorem of Mahler states that any continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ may be written uniquely in the form

$$f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n},$$

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where $a_n(f) \in \mathbb{Q}_p, a_n(f) \mapsto 0$ as $n \mapsto \infty$. In fact

$$a_n(f) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(j). \tag{1.1}$$

This theorem may be generalized to continuous functions $f : \mathbb{Z}_p \to K$, where K is any finite extension of $\mathbb{Q}_p$. Using this generalization, we obtain the following

$$\int_{\mathbb{Z}_p} f(x) d\alpha(x) = \sum_{n=0}^{\infty} a_n(f) \int_{\mathbb{Z}_p} \binom{x}{n} d\alpha(x) = \sum_{n=0}^{\infty} a_n(f) b_n(\alpha).$$

Note that if $\mathcal{O}$ is the ring of integers of K and $f : \mathbb{Z}_p \mapsto \mathcal{O}$, then $a_n(f) \in \mathcal{O}$.

For $a \in \mathbb{Z}_p^\times$, denote by $\alpha \circ a$ the measure on $\mathbb{Z}_p$ given by $\alpha \circ a(A) = \alpha(aA)$ for all compact open subsets A of $\mathbb{Z}_p$. Also, for a compact open subset $A \subseteq \mathbb{Z}_p$, we let $\alpha|_A$ denote the measure obtained by restricting α to A and extending by 0.

The Γ-transform of a measure α is defined as a function of the p-adic variable s given by

$$\Gamma_\alpha(s) = \int_{\mathbb{Z}_p^\times} < x >^s d\alpha(x).$$

Splitting up the integral, and putting $d\alpha(ax)$ for $d\alpha \circ a(x)$, we can also write

$$\Gamma_\alpha(s) = \sum_{\eta \in V} \int_U < \eta x >^s d\alpha(\eta x) = \int_U x^s d\beta(x),$$

where

$$\beta = \sum_{\eta \in V} (\alpha \circ \eta)|_U,$$

a measure on U.

Now the measure β may be viewed as a measure on $\mathbb{Z}_p$ via the isomorphism ϕ:

$$\tilde{\beta}(A) = \beta(\phi(A)).$$

It is customary to write $d\beta(u^y)$ for $d\tilde{\beta}(y)$. Let $G(T)$ be the power series associated to $\tilde{\beta}$, that is,

$$G(T) = \int_{\mathbb{Z}_p} T^y d\beta(u^y).$$

Then $\Gamma_\alpha(s) = G(u^s)$, so that $\Gamma_\alpha(s)$ is an Iwasawa function over $\mathcal{O}$.

2. Iwasawa λ-invariants and Γ- transforms

The Iwasawa μ and λ- invariants of a power series

$$F(T) = \sum_{n=0}^{\infty} a_n (T-1)^n \in \mathcal{O}[[T-1]]$$

are defined by

$$\mu(F(T)) = \min\{ord(a_n) : n \geq 0\}$$
$$\lambda(F(T)) = \min\{n : ord(a_n) = \mu(F(T))\}$$

For a measure α, we understand $\mu(\alpha)$ and $\lambda(\alpha)$ to mean $\mu(\hat{\alpha}(T))$ and $\lambda(\hat{\alpha}(T))$.

Let $\alpha \in \Lambda$ be a $\mathcal{O}$-valued measures on $\mathbb{Z}_p$. Let u be a fixed topological generator of $U = 1 + p\mathbb{Z}_p$, and let $G(T)$ satisfy $G(u^s) = \Gamma_\alpha(s)$, so that

$$G(T) = \int_{\mathbb{Z}_p} T^y d\beta(u^y), \text{ where } \beta = \sum_{\eta \in V} (\alpha \circ \eta)|_U. \tag{2.1}$$

Note that β is a measure on U. We extend β to $\mathbb{Z}_p$ by 0 and then we get a power series $\hat{\beta}(T) = \sum_{n=0}^{\infty} b_n (T-1)^n$. Suppose that $G(T) = \sum_{n=0}^{\infty} g_n (T-1)^n$. Sinnott in his paper [4] proved that $\mu(G(T)) = \mu(\alpha^* + \alpha^* \circ (-1))$, if $\hat{\alpha}(T)$ is a rational function of T. Here $\alpha^* = \alpha|_{\mathbb{Z}_p^\times}$. It was Kida who first obtained a relation between the λ-invariant of a measure and its Gamma-Transform with a fixed topological generator [2]. Later, Nancy Childress proved the following results in her paper [1]:

Result 2.1. *$\mu(G(T)) = \mu(\beta)$.*

Result 2.2. *Suppose $\lambda(G(T)) \leq p$, then $\lambda(\beta) = p\lambda(G(T))$.*

She remarked that it would be interesting to know whether her methods can be extended for larger $\lambda(G(T))$. Satoh obtained the same result without any condition on $\lambda(G(T))$, but his approach was based on certain properties of Stirling numbers [3]. In this paper we prove the following main result in the spirit of Childress.

Theorem 2.3. *Suppose $\lambda(G(T)) \leq 2p$, then $\lambda(\beta) = p\lambda(G(T))$.*

We will prove this theorem exploiting certain combinatorial identities, which we shall prove in the next section. Through our approach we also derive certain p-adic properties of Mahler coefficients. Note that the relation between b_m and g_m is given by the following result in Childress [1].

Result 2.4. *If $n \geq ord_p(m!)$, then $b_m \equiv \sum_{r=0}^{n} g_r a_r(f_m) \pmod{p}$.*

Here, $a_m(f_n)$s are the Mahler coefficients of $f_n(x) = \binom{u^x}{n} = \sum_{m=0}^{\infty} a_m(f_n)\binom{x}{m}$. We will investigate p-adic properties of the Mahler coefficients $a_m(f_n)$. In order to study the Mahler coefficients $a_m(f_n)$ we will require certain identities involving binomial coefficients, which will be established in a combinatorial fashion in the next section.

3. Certain Combinatorial Identities

The following result was a crucial ingredient in the work of Childress [1].

Result 3.1.

$$\sum_{i=1}^{n} (-1)^{n-i} \binom{n}{i} \binom{ti}{n} = t^n.$$

Here we will prove a more general result.

Lemma 3.2. *For non-negative integers n, t, k, we have*

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{t(i+k)}{n} = t^n. \tag{3.1}$$

Proof. The result is obvious for $t = 0$ or $n = 0$. So we assume $n, t \geq 1$ and $k \geq 0$. Let N, N', T be sets such that $N \subseteq N', |N| = n, |N'| = n + k$, and $|T| = t$. Let R be the set of all n-subsets of $N' \times T$. Clearly $|R| = \binom{t(n+k)}{n}$. Also, for $a \in N$, let R_a be the set of all n-subsets A of $N' \times T$ such that $(a, b) \notin A$ for any $b \in T$. Obviously R_a is the set of all n-subsets of $(N' - \{a\}) \times T$ and hence $|R_a| = \binom{t(n+k-1)}{n}$.

For $I \subseteq N$, let R_I be the set of all n-subsets A of $N' \times T$ such that $(a, b) \notin A$ for any $a \in I$ and for any $b \in T$. Clearly R_I is the set of all n-subsets of $(N' - I) \times T$ and hence

$$|R_I| = \binom{t(n+k-i)}{n}, \text{ where } |I| = i. \tag{3.2}$$

If $I = \{a_1, \cdots, a_i\}$, then clearly $R_I = R_{a_1} \cap \cdots \cap R_{a_i}$. Thus $|R_{a_1} \cap \cdots \cap R_{a_i}| = \binom{t(n+k-i)}{n}$. By inclusion-exclusion principle, we get

$$\begin{aligned} |\bigcup_{a \in N} R_a| &= \sum_{a \in N} |R_a| - \sum_{\{a_1, a_2\} \subseteq N} |R_{a_1} \cap R_{a_2}| + \cdots + (-1)^{i+1} \sum_{\{a_1, \cdots, a_i\} \subseteq N} |R_{a_1} \cap \cdots \cap R_{a_i}| \\ &\quad + \cdots + (-1)^{n+1} |\bigcap_{a \in N} R_a| \\ &= \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \binom{(n+k-i)t}{n}. \end{aligned} \tag{3.3}$$

Therefore,

$$\begin{aligned} |R - \bigcup_{a \in N} R_a| = |R| - |\bigcup_{a \in N} R_a| &= \binom{t(n+k)}{n} - \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \binom{(n+k-i)t}{n} \\ &= \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} \binom{t(n+k-i)}{n} \\ &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{t(i+k)}{n}. \end{aligned} \tag{3.4}$$

A function $f : N \to T$ may be viewed as an n-subset of $N \times T$. Conversely, an n-subset $A \subseteq N \times T$ defines a function $f : N \to T$ if and only if the cardinality of the set $\{a \in N : (a, b) \in A \text{ for some } b \in T\}$ is equal to n. Therefore, it is not difficult to see that there is a one-to-one correspondence between $R - \bigcup_{a \in N} R_a$ and the set of all functions from N to T. Thus $|R - \bigcup_{a \in N} R_a| = t^n$, which proves the result because of (3.4). □

Remark 3.3. *The result* (3.1) *of Childress is nothing but lemma* (3.2) *with* $k = 0$.

Lemma 3.4. *For non-negative integers* n, t *with* $n > 1$, *we have*

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{ti}{n-1} = 0.$$

Proof: Since $n>1$, we have $\binom{n}{i}=\binom{n-1}{i}+\binom{n-1}{i-1}$. Using this and Lemma (3.2) for $k=1$, we get

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\binom{ti}{n-1}\\
&=\left\{\sum_{i=0}^{n}(-1)^{n-i}\binom{n-1}{i}\binom{ti}{n-1}\right\}+\left\{\sum_{i=0}^{n}(-1)^{n-i}\binom{n-1}{i-1}\binom{ti}{n-1}\right\}\\
&=-\left\{\sum_{i=0}^{n-1}(-1)^{n-1-i}\binom{n-1}{i}\binom{ti}{n-1}\right\}+\left\{\sum_{i=0}^{n-1}(-1)^{n-1-i}\binom{n-1}{i}\binom{t(i+1)}{n-1}\right\}\\
&=-t^{n-1}+t^{n-1}\\
&=0.
\end{aligned}$$

4. p-adic properties of Mahler coefficients $a_m(f_n)$

Let us fix a topological generator $u=1+t_1p+t_2p^2+\cdots$ of $1+p\mathbb{Z}_p$. Hence t_1 is a unit. It is not difficult to see that

$$(1+T)^{u^{p+n}}\equiv(1+T)(1+T^p)^{nt_1}(1+T^{p^2})^{t_1+\frac{n(n-1)}{2}t_1^2+nt_2}+\text{ higher order terms} \pmod p. \tag{4.1}$$

$$(1+T)^{u^{n}}\equiv(1+T)(1+T^p)^{nt_1}(1+T^{p^2})^{\frac{n(n-1)}{2}t_1^2+nt_2}+\text{ higher order terms} \pmod p. \tag{4.2}$$

Using these binomial expansions, we prove the following lemmas about the Mahler coefficients $a_m(f_n)$ for different m and n.

Lemma 4.1. *Suppose that $1\le k<p$ and $p^2+(k-1)p\le m<p^2+kp$. Then*

$$a_{p+k}(f_m)\equiv 0 \pmod p.$$

Proof: From (1.1), we have

$$a_{p+k}(f_m)=\sum_{j=0}^{p+k}(-1)^{p+k-j}\binom{p+k}{j}\binom{u^j}{m}. \tag{4.3}$$

But, $\binom{u^j}{m}$ is the co-efficient of T^m in the expansion of $(1+T)^{u^j}$. Clearly, if $p^2+(k-1)p\le m<p^2+kp$ and $m\neq p^2+(k-1)p, p^2+(k-1)p+1$, then from (4.1) and (4.2) we find that the co-efficient of T^m in $(1+T)^{u^j}$ is zero modulo p. Also, co-efficients of T^m modulo p in $(1+T)^{u^j}$ are equal for $m=p^2+(k-1)p, p^2+(k-1)p+1$. Thus, to prove that $a_{p+k}(f_m)$ is zero modulo p when $m=p^2+(k-1)p, p^2+(k-1)p+1$, we need to prove for $m=p^2+(k-1)p$ only. If $k=1$, then

$$a_{p+1}(f_{p^2})\equiv-\binom{u}{p^2}-\binom{u^p}{p^2}+\binom{u^{p+1}}{p^2}\equiv -t_2-t_1+(t_1+t_2)\equiv 0 \pmod p. \tag{4.4}$$

Therefore, we assume that $k > 1$. From (4.1) and (4.2), we have

$$\begin{aligned} \binom{u^j}{m} &= \text{co-efficient of } T^m \text{ in the expansion of } (1+T)^{u^j} \\ &\equiv \binom{jt_1}{k-1}\left\{\frac{j(j-1)}{2}t_1^2 + jt_2\right\} \pmod{p} \;\; if \;\; j < p \end{aligned} \tag{4.5}$$

and

$$\binom{u^j}{m} \equiv \binom{it_1}{k-1}\left\{t_1 + \frac{i(i-1)}{2}t_1^2 + it_2\right\} \pmod{p} \;\; if \;\; j = p+i, 0 \le i < p. \tag{4.6}$$

Now,

$$\begin{aligned} a_{p+k}(f_m) &= \sum_{j=0}^{p+k}(-1)^{p+k-j}\binom{p+k}{j}\binom{u^j}{m} \\ &\equiv \sum_{j=0}^{k}(-1)^{p+k-j}\binom{p+k}{j}\binom{jt_1}{k-1}\left\{\frac{j(j-1)}{2}t_1^2 + jt_2\right\} \\ &\quad + \sum_{j=p}^{p+k}(-1)^{p+k-j}\binom{p+k}{j}\binom{u^j}{m} \\ &\equiv -\sum_{j=0}^{k}(-1)^{k-j}\binom{p+k}{j}\binom{jt_1}{k-1}\left\{\frac{j(j-1)}{2}t_1^2 + jt_2\right\} \\ &\quad + \sum_{j=0}^{k}(-1)^{k-j}\binom{p+k}{k-j}\binom{jt_1}{k-1}\left\{t_1 + \frac{j(j-1)}{2}t_1^2 + jt_2\right\} \pmod{p}. \end{aligned} \tag{4.7}$$

Again, $\binom{p+k}{j} \equiv \binom{k}{j} \pmod{p}$ and hence (4.7) implies that

$$a_{p+k}(f_m) \equiv \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{jt_1}{k-1}t_1 \pmod{p}. \tag{4.8}$$

Using Lemma (3.4), we complete the proof of $a_{p+k}(f_m) \equiv 0 \pmod{p}$ when $m = p^2 + (k-1)p$ and this completes the proof of the lemma.

Lemma 4.2. *Suppose that* $1 \le k < p$. *Then*

$$a_{p+k}(f_{p^2+kp}) \equiv t_1^{k+1} \pmod{p} \;\; and \;\; a_{p+k+1}(f_{p^2+kp}) \equiv 0 \pmod{p}.$$

Proof: Proceeding as Lemma (4.1), we find that

$$a_{p+k}(f_{p^2+kp}) \equiv t_1 \times \left\{\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\binom{jt_1}{k}\right\} \pmod{p}$$

$$\text{and} \quad a_{p+k+1}(f_{p^2+kp}) \equiv t_1 \times \left\{\sum_{j=0}^{k+1}(-1)^{k+1-j}\binom{k+1}{j}\binom{jt_1}{k}\right\} \pmod{p}.$$

Using result (3.1) and lemma (3.4), we complete the proof of the lemma.

Lemma 4.3. *Suppose that* $2p^2 - p \le m < 2p^2$. *Then* $a_{2p}(f_m) \equiv 0 \pmod p$. *Also,*

$$a_{2p}(f_{2p^2}) \equiv t_1^2 \pmod p, \quad a_{2p+1}(f_{2p^2}) \equiv 0 \pmod p, \quad \textit{and } a_{2p+2}(f_{2p^2}) \equiv 0 \pmod p.$$

Proof: Suppose that $2p^2 - p \le m < 2p^2$. From (1.1), we have

$$\begin{aligned} a_{2p}(f_m) &= \sum_{j=0}^{2p} (-1)^{2p-j} \binom{2p}{j} \binom{u^j}{m} \\ &\equiv -\binom{2p}{p}\binom{u^p}{m} + \binom{u^{2p}}{m} \\ &\equiv \text{co-efficient of } T^m \text{ in } \left\{ -\binom{2p}{p} \times (1+T)^{u^p} + (1+T)^{u^{2p}} \right\} \\ &\equiv 0 \pmod p. \end{aligned} \tag{4.9}$$

We obtain (4.9) using the binomial expansion (4.1).

Again,

$$\begin{aligned} a_{2p}(f_{2p^2}) &\equiv -\binom{2p}{p}\binom{u^p}{2p^2} + \binom{u^{2p}}{2p^2} \\ &\equiv \text{co-efficient of } T^{2p^2} \text{ in } \left\{ -\binom{2p}{p} \times (1+T)^{u^p} + (1+T)^{u^{2p}} \right\} \\ &\equiv -2\binom{t_1}{2} + \binom{2t_1}{2} \\ &\equiv t_1^2 \pmod p. \end{aligned} \tag{4.10}$$

Also, modulo p

$$\begin{aligned} a_{2p+1}(f_{2p^2}) &\equiv \binom{u}{2p^2} + \binom{2p+1}{p}\left\{ \binom{u^p}{2p^2} - \binom{u^{p+1}}{2p^2} \right\} - \binom{u^{2p}}{2p^2} + \binom{u^{2p+1}}{2p^2} \\ &\equiv \text{co-efficient of } T^{2p^2} \text{ in } (1+T)^u + \binom{2p+1}{p}\left\{ (1+T)^{u^p} - (1+T)^{u^{p+1}} \right\} \\ &\qquad - (1+T)^{u^{2p}} + (1+T)^{u^{2p+1}} \\ &\equiv \binom{t_2}{2} + \binom{2p+1}{p}\left\{ \binom{t_1}{2} - \binom{t_1+t_2}{2} \right\} - \binom{2t_1}{2} + \binom{2t_1+t_2}{2}. \end{aligned} \tag{4.11}$$

But, $\binom{2p+1}{p} \equiv 2 \pmod p$. Using this in (4.11), we find that

$$a_{2p+1}(f_{2p^2}) \equiv 0 (\text{mod } p). \tag{4.12}$$

Finally, we prove that $a_{2p+2}(f_{2p^2}) \equiv 0 \pmod p$.

Using $\binom{2p+2}{p} \equiv 2 \pmod{p}$ and $\binom{2p+2}{p+1} \equiv 4 \pmod{p}$, we find that

$$
\begin{aligned}
a_{2p+2}(f_{2p^2}) &\equiv -2\binom{u}{2p^2} + \binom{u^2}{2p^2} - 2\binom{u^p}{2p^2} + 4\binom{u^{p+1}}{2p^2} \\
&\quad - 2\binom{u^{p+2}}{2p^2} + \binom{u^{2p}}{2p^2} - 2\binom{u^{2p+1}}{2p^2} + \binom{u^{2p+2}}{2p^2} \\
&\equiv \text{co-efficient of } T^{2p^2} \text{ in } -2(1+T)^u + (1+T)^{u^2} - 2(1+T)^{u^p} + 4(1+T)^{u^{p+1}} \\
&\quad - 2(1+T)^{u^{p+2}} + (1+T)^{u^{2p}} - 2(1+T)^{u^{2p+1}} + (1+T)^{u^{2p+2}} \\
&\equiv -2\binom{t_2}{2} + \binom{t_1^2+2t_2}{2} - 2\binom{t_1}{2} + 4\binom{t_1+t_2}{2} - 2\binom{t_1^2+t_1+2t_2}{2} \\
&\quad + \binom{2t_1}{2} - 2\binom{2t_1+t_2}{2} + \binom{t_1^2+2t_1+2t_2}{2} \\
&\equiv 0 \pmod{p}. \qquad (4.13)
\end{aligned}
$$

This completes the proof of the lemma.

5. **Proof of Main Result**

Now we have all the ingredients for the proof of the main result. We may assume that $\mu(G(T)) = 0$, because $\mu(G(T)) = \mu(\beta)$ by result (2.1), and for any power series $F(T) \in \mathcal{O}[[T-1]]$, if $\pi | F(T)$ then $\lambda(\pi^{-1}F(T)) = \lambda(F(T))$. Childress in her paper [1] proved that if $\lambda(G(T)) \leq p$, then $\lambda(\beta) = p\lambda(G(T))$. Hence it is enough to prove the Theorem (2.3) for $p < \lambda(G(T)) \leq 2p$.

Case (i): Suppose that $\lambda(G) = p + k$ where $0 < k < p$. Then $g_i \equiv 0 \pmod{\pi}$ for $i = 0, \cdots, p+k-1$ and g_{p+k} is a unit. Clearly, $\text{ord}_p((p^2+kp)!) = p+k+1$ and if $m < p^2 + kp$, then $\text{ord}_p(m!) \leq p+k$. Also, if $m < p^2 + (k-1)p$, then $\text{ord}_p(m!) < p+k$. Using result (2.4) and $g_i \equiv 0 \pmod{\pi}$ for $i = 0, \cdots, p+k-1$, we have

$$b_m \equiv 0 (\text{mod } \pi) \text{ if } m < p^2 + (k-1)p \qquad (5.1)$$

and

$$b_m \equiv g_{p+k} a_{p+k}(f_m) (\text{mod } \pi) \text{ if } p^2 + (k-1)p \leq m < p^2 + kp. \qquad (5.2)$$

¿From lemma (4.1) and (5.2), we get $b_m \equiv 0 \pmod{\pi}$ and hence

$$b_m \equiv 0 (\text{mod } \pi) \text{ if } m < p^2 + kp. \qquad (5.3)$$

Since $\text{ord}_p((p^2+kp)!) = p+k+1$, using Lemma (4.2), we have

$$
\begin{aligned}
b_{p^2+kp} &\equiv \sum_{r=0}^{p+k+1} g_r a_r(f_{p^2+kp}) \pmod{p} \\
&\equiv g_{p+k} a_{p+k}(f_{p^2+kp}) + g_{p+k+1} a_{p+k+1}(f_{p^2+kp}) \pmod{\pi} \\
&\equiv g_{p+k} t_1^{k+1} \pmod{\pi}, \qquad (5.4)
\end{aligned}
$$

which is a unit in $\mathcal{O}$. This proves that $\lambda(\beta) = p^2 + kp = p\lambda(G(T))$.

Case (ii): Now suppose that $\lambda(G(T)) = 2p$. Then $g_i \equiv 0 \pmod{\pi}$ for $i = 0, \cdots, 2p-1$ and g_{2p} is a unit in $\mathcal{O}$. If $m < 2p^2 - p$, then $\text{ord}_p(m!) < 2p$ and hence from result (2.4), we have $b_m \equiv 0 \pmod{\pi}$. If $2p^2 - p \leq m < 2p^2$, then $\mathrm{ord}_p(m!) \leq 2p$ and hence from result (2.4) and lemma (4.3), we have

$$b_m \equiv \sum_{r=0}^{2p} g_r a_r(f_m) \pmod{p} \equiv g_{2p} a_{2p}(f_m) \equiv 0 \pmod{\pi}. \tag{5.5}$$

Thus, if $m < 2p^2$, then $b_m \equiv 0 \pmod{\pi}$. Again, $\mathrm{ord}_p((2p^2)!) = 2p + 2$ and hence

$$\begin{aligned} b_{2p^2} &\equiv \sum_{r=0}^{2p+2} g_r a_r(f_m) \pmod{p} \\ &\equiv g_{2p} a_{2p}(f_{2p^2}) + g_{2p+1} a_{2p+1}(f_{2p^2}) + g_{2p+2} a_{2p+2}(f_{2p^2}) \pmod{\pi}. \end{aligned} \tag{5.6}$$

¿From (4.10), (4.12), (4.13), and (5.6), we have $b_{2p^2} \equiv g_{2p} t_1^2 \pmod{\pi}$. Therefore, b_{2p^2} is a unit in $\mathcal{O}$ and hence $\lambda(\beta) = 2p^2 = p\lambda(G(T))$. This completes the proof of the main theorem.

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