IWASAWA $\lambda$-INVARIANTS OF $p$-ADIC MEASURES ON $\mathbb{Z}_p^n$ AND THEIR $\Gamma$-TRANSFORMS

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Abstract. In [2], we proved a relation between the $\lambda$-invariants of a $p$-adic measure on $\mathbb{Z}_p^n$ and its $\Gamma$-transform under a strong condition. In this paper, we determine the relation without imposing any condition. We also determine $p$-adic properties of certain Mahler coefficients by exploiting some combinatorial identities.

Key Words: $p$-adic measure, $\Gamma$-transform, Iwasawa invariants, Mahler coefficients.

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1. Introduction

The theory of $\Gamma$-transform is very useful in studying the Iwasawa invariants of imaginary abelian number fields. In [8], Sinnott gave an elegant new proof of the theorem of Ferrero and Washington that the Iwasawa $\mu$-invariant is zero for the cyclotomic $\mathbb{Z}_p$-extension of any abelian number field. Sinnott further showed how to compute the $\mu$-invariant of the $\Gamma$-transform of a rational function. In [4], Katz showed that the $p$-adic $L$-functions of a totally real number field $K$ do indeed arise from roughly rational function measures on $\mathbb{Z}_p^d$, where $d = [K : \mathbb{Q}]$.

It was Kida who first obtained a relation between the $\lambda$-invariant of a measure on $\mathbb{Z}_p$ and its $\Gamma$-transform [5]. In our paper [2], exploiting certain combinatorial identities we determined a relation between the $\lambda$-invariants of a $p$-adic measure on $\mathbb{Z}_p^n$ and its $\Gamma$-transform for any $n \geq 1$ under some restrictive hypothesis. In this paper, we generalize the results of [2].

Let $p$ be a fixed odd prime. We have $\mathbb{Z}_p^\times = V \times U$, where $V$ is the group of $(p - 1)$st roots of unity in $\mathbb{Z}_p$ and $U = 1 + p\mathbb{Z}_p$. Then the projections from $\mathbb{Z}_p^\times$ onto $V$ and $U$ are denoted by $\omega$ and $< >$, respectively. By $u$, we will denote a fixed topological generator of $U$. Then there is an isomorphism $\phi : \mathbb{Z}_p \to U$ given by $\phi(y) = u^y$. Let $n \geq 1$. By fixing a topological generator $u_i$ ($1 \leq i \leq n$) for each copy of $\mathbb{Z}_p$ in $\mathbb{Z}_p^n$, we obtain an isomorphism $\phi^n : \mathbb{Z}_p^n \to U^n$ given by $(y_1, \ldots, y_n) \mapsto (u_1^{y_1}, \ldots, u_n^{y_n})$.

Let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$ with a local parameter $\pi$. It is well-known that there is an isomorphism between the ring $\Lambda_n$ of $\mathcal{O}$-valued measures on $\mathbb{Z}_p^n$ under convolution and the power series ring $\mathcal{O}[[T_1 - 1, \ldots, T_n - 1]]$. Explicitly, for $x \in \mathbb{Z}_p$, if we put

$$T^x = \sum_{m=0}^{\infty} \binom{x}{m} (T - 1)^m \in \mathcal{O}[[T - 1]],$$

then

$$\mathcal{O}[[T - 1]] \ni \phi^n(y_1, \ldots, y_n) = \sum_{m=0}^{\infty} \binom{y_1 + \ldots + y_n}{m} (\text{Tr } T)^m.$$
then the unique power series $\hat{\alpha}(T_1, \ldots, T_n)$ associated with $\alpha$ is given by

$$
\hat{\alpha}(T_1, \ldots, T_n) = \int_{\mathbb{Z}_p^n} T_1^{x_1} \cdots T_n^{x_n} d\alpha(x_1, \ldots, x_n)
= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \left( \int_{\mathbb{Z}_p^{m_1}} (x_1) \cdots \int_{\mathbb{Z}_p^{m_n}} (x_n) d\alpha(x_1, \ldots, x_n) \right) 
\times (T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}.
$$

(1.1)

If $a = (a_1, \ldots, a_n) \in (\mathbb{Z}_p^\times)^n$, we denote by $\alpha \circ a$ the measure on $\mathbb{Z}_p^n$ given by $\alpha \circ a(A_1 \times \cdots \times A_n) = \alpha(a_1 A_1 \times \cdots \times a_n A_n)$, where $A_i$ are compact open subsets of $\mathbb{Z}_p$. Also, for compact open subsets $A_i$ of $\mathbb{Z}_p$, we let $\alpha|_A$ denote the measure obtained by restricting $\alpha$ to $A$ and extending by 0, where $A = A_1 \times \cdots \times A_n$.

The $\Gamma$-transform of $\alpha$ is defined as a function of the $p$-adic variables $s_1, \ldots, s_n$ given by

$$
\Gamma_\alpha(s_1, \ldots, s_n) = \int_{(\mathbb{Z}_p^\times)^n} < x_1 >^{s_1} \cdots < x_n >^{s_n} d\alpha(x_1, \ldots, x_n)
= \sum_{\eta_1 \in V} \cdots \sum_{\eta_n \in V} \int_{U^n} < \eta_1 x_1 >^{s_1} \cdots < \eta_n x_n >^{s_n} d\alpha(\eta_1 x_1, \ldots, \eta_n x_n)
= \int_{U^n} x_1^{s_1} \cdots x_n^{s_n} d\beta(x_1, \ldots, x_n),
$$

where $d\alpha(\eta_1 x_1, \ldots, \eta_n x_n)$ denotes $d(\alpha \circ (\eta_1, \ldots, \eta_n))(x_1, \ldots, x_n)$ and

$$
\beta = \sum_{\eta_1 \in V} \cdots \sum_{\eta_n \in V} (\alpha \circ (\eta_1, \ldots, \eta_n))|_{U^n},
$$

(1.2)

a measure on $U^n$. By the isomorphism $\phi^n : \mathbb{Z}_p^n \rightarrow U^n$, one can transport the measure $\beta$ on $U^n$ to a measure $\tilde{\beta}$ on $\mathbb{Z}_p^n$. It is clear from (1.1) that the power series $\tilde{\beta}$ associated with $\tilde{\beta}$ interpolates the $\Gamma$-transform of $\alpha$ as

$$
\Gamma_\alpha(s_1, \ldots, s_n) = \int_{\mathbb{Z}_p^n} (u_1^{s_1})^{y_1} \cdots (u_n^{s_n})^{y_n} d\tilde{\beta}(y_1, \ldots, y_n) = \tilde{\beta}(u_1^{s_1}, \ldots, u_n^{s_n}).
$$

(1.3)

The measure $\beta$ on $U^n$ can be extended by 0 to $\mathbb{Z}_p^n$, and the associated power series of the extended measure will be denoted by $\tilde{\beta}$.

2. Iwasawa Invariants

The Iwasawa $\mu$- and $\lambda$- invariants of a power series

$$
F(T) = \sum_{n=0}^{\infty} a_n (T - 1)^n \in \mathcal{O}[[T - 1]]
$$


are defined by
\[
\mu(F(T)) = \min \{ \text{ord}_\pi(a_n) : n \geq 0 \}
\]
\[
\lambda(F(T)) = \min \{ n : \text{ord}_\pi(a_n) = \mu(F(T)) \}.
\]
Analogously, in [2] we defined the Iwasawa \(\mu\)- and \(\lambda\)- invariants of a power series
\[
F(T_1, \ldots, T_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} a_{m_1, \ldots, m_n} (T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}
\]
in \(O[[T_1 - 1, \ldots, T_n - 1]]\) as follows:
\[
\mu(F(T_1, \ldots, T_n)) = \min \{ \text{ord}_\pi(a_{m_1, \ldots, m_n}) : m_i \geq 0 \ \forall i \}
\]
\[
\lambda(F(T_1, \ldots, T_n)) = \min \{ m_1 + \cdots + m_n : \text{ord}_\pi(a_{m_1, \ldots, m_n}) = \mu(F(T_1, \ldots, T_n)) \}.
\]

**Definition 1.** Let \(\alpha \in \Lambda_n\). The Iwasawa \(\mu\)- and \(\lambda\)- invariants of \(\alpha\) are defined as \(\mu(\hat{\alpha}(T_1, \ldots, T_n))\) and \(\lambda(\hat{\alpha}(T_1, \ldots, T_n))\) respectively. Similarly, the Iwasawa invariants of \(\Gamma_\alpha\) are defined as the corresponding invariants of the power series \(\hat{\beta}(T_1, \ldots, T_n)\).

Let \(\alpha \in \Lambda_n\). In case of \(n = 1\), Sinnott in his paper [8] proved that \(\mu(\Gamma_\alpha) = \mu(\alpha^* + \alpha^* \circ (-1))\), if \(\hat{\alpha}(T)\) is a rational function of \(T\). Here \(\alpha^* = \alpha|_{\mathbb{Z}_p^\times}\). It is known that \(\mu(\Gamma_\alpha) = \mu(\beta)\) (see for example [3, 8]). It is easy to prove that \(\mu(\Gamma_\alpha) = \mu(\beta)\) for any \(n \geq 1\) (see [2, Lemma 2.2]). It would be interesting to extend it to study \(\lambda\)-invariants.

The case \(n = 1\) has been studied in [3, 5, 6, 7]. The aim of this paper is to prove the following main result.

**Theorem 2.1.** Let \(\alpha\) be an \(O\)-valued measure on \(\mathbb{Z}_p^n\). Define a measure \(\beta\) on \(U^n\) by (1.2) and let \(\hat{\beta}(T_1, \ldots, T_n)\) be the power series associated with the measure \(\beta\) on \(U^n\) extended to \(\mathbb{Z}_p^n\) by zero. Then \(\lambda(\beta) = p\lambda(\Gamma_\alpha)\).

**Remark 2.2.** The above theorem was proved under some restrictive hypothesis in [2, Lemma 2.2].

3. Mahler coefficients and proof of the Theorem 2.1

A crucial ingredient in our proof of theorem 2.1 is a relation (see lemma 3.1) between coefficients of the power series \(\hat{\beta}\) and \(\hat{\beta}\) via Mahler coefficients. A classical theorem of Mahler states that any continuous function \(f : \mathbb{Z}_p \to O\) can be written uniquely in the form
\[
f(x) = \sum_{j=0}^{\infty} a_j(f) \left( \begin{array}{c} x \\ j \end{array} \right),
\]
where \(a_j(f) \in O, a_j(f) \to 0\) as \(j \to \infty\). In fact
\[
a_j(f) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} f(i).
\]
Furthermore, if \( f : \mathbb{Z}_p^n \to \mathcal{O} \) is continuous, we may write (by repeated application of (3.1))

\[
f(x_1, \ldots, x_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} a_{m_1, \ldots, m_n}(f) \left( \frac{x_1}{m_1} \right) \cdots \left( \frac{x_n}{m_n} \right),
\]

where

\[
a_{m_1, \ldots, m_n}(f) = \sum_{j_1=0}^{m_1} \cdots \sum_{j_n=0}^{m_n} (-1)^{m_1-j_1} \cdots (-1)^{m_n-j_n} \binom{m_1}{j_1} \cdots \binom{m_n}{j_n}
\]

\[
\times f(j_1, \ldots, j_n) \in \mathcal{O}.
\]

The constants \( a_{m_1, \ldots, m_n}(f) \) are called the Mahler coefficients of the function \( f \).

Let us consider the continuous functions \( f_m : \mathbb{Z}_p \to \mathbb{Z}_p \) and \( f_{m_1, \ldots, m_n} : \mathbb{Z}_p^n \to \mathbb{Z}_p \) defined by

\[
f_m(x) = \left( \frac{u^x}{m} \right) \quad \text{and} \quad f_{m_1, \ldots, m_n}(x_1, \ldots, x_n) = f_{m_1}(x_1) \cdots f_{m_n}(x_n).
\]

Now, if \( a_m(f_k) \) are the Mahler coefficients of \( f_k(x) = \left( \frac{u^x}{k} \right) = \sum_{m=0}^{\infty} a_m(f_k) \left( \frac{x}{m} \right) \), then

\[
a_{j_1, \ldots, j_n}(f_{m_1, \ldots, m_n}) = a_{j_1}(f_{m_1}) \cdots a_{j_n}(f_{m_n}). \tag{3.4}
\]

Suppose

\[
\hat{\beta}(T_1, \ldots, T_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} b_{m_1, \ldots, m_n}(T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}
\]

and

\[
\hat{\beta}(T_1, \ldots, T_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} g_{m_1, \ldots, m_n}(T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}.
\]

Then we have the following important lemma (see lemma 2.3 in [2]) which relates the coefficients of \( \hat{\beta}(T_1, \ldots, T_n) \), \( \hat{\beta}(T_1, \ldots, T_n) \), and certain Mahler coefficients.

**Lemma 3.1.** *Modulo \( p^{n+k_1+\cdots+k_n}\mathcal{O} \), we have*

\[
m_1! \cdots m_n! b_{m_1, \ldots, m_n} \equiv m_1! \cdots m_n! \sum_{j_1=0}^{k_1} \cdots \sum_{j_n=0}^{k_n} g_{j_1, \ldots, j_n} a_{j_1, \ldots, j_n}(f_{m_1, \ldots, m_n}),
\]

*where \( a_{j_1, \ldots, j_n}(f_{m_1, \ldots, m_n}) \) are the Mahler coefficients of \( f_{m_1, \ldots, m_n}(x_1, \ldots, x_n) \).*

Note that when \( \text{ord}_p(m_1! \cdots m_n!) \leq k_1 + \cdots + k_n \), then

\[
b_{m_1, \ldots, m_n} \equiv \sum_{j_1=0}^{k_1} \cdots \sum_{j_n=0}^{k_n} g_{j_1, \ldots, j_n} a_{j_1, \ldots, j_n}(f_{m_1, \ldots, m_n}) \pmod{p^n\mathcal{O}}. \tag{3.5}
\]

In order to prove the theorem 2.1, we need to investigate \( p \)-adic properties of the Mahler coefficients \( a_{j_1, \ldots, j_n}(f_{m_1, \ldots, m_n}) \). We shall now study these coefficients using certain combinatorial identities.

Let us fix a topological generator \( u = 1 + t_1 p + t_2 p^2 + \cdots \) of \( 1 + p\mathbb{Z}_p \). Hence \( t_1^k \) is a unit. In fact \( t_1 \) is an integer lying between 1 and \( p - 1 \). We now state a binomial
expansion in the following lemma. One can find a simple proof using the fact that $(1 + T)^{p^i} \equiv (1 + T^{p^i}) \pmod{p}$ for $i \geq 1$.

**Lemma 3.2.** For $n \geq 1$, we have

$$(1 + T)^n \equiv (1 + T)(1 + T^p)^{nt_1}(1 + T^{p^2})^{a_{n,2} + nt_2} \cdots (1 + T^{p^i})^{a_{n,j} + nt_j} \cdots \pmod{p},$$

(3.6)

where $a_{n,j} \geq 0$ for all $j \geq 2$.

In the following lemma, we prove another binomial expansion.

**Lemma 3.3.** For $k \geq 1$, let $m = l_k p^k + l_{k-1} p^{k-1} + \cdots + l_1 p + l_0$, where $0 \leq l_i < p$ for all $i = 0, 1, \ldots, k$. Then we have

$$(1 + T)^m \equiv (1 + T)(1 + T^p)^{l_1 t_1}(1 + T^{p^2})^{l_1 t_1 + l_0 t_2 + a_{m,2}}(1 + T^{p^3})^{l_2 t_1 + l_1 t_2 + l_0 t_3 + a_{m,3}} \cdots
$$

$$(1 + T^{p^j})^{l_j t_1 + \cdots + l_{j-1} t_{j-1} + l_1 t_{j-1} + l_0 t_{j} + a_{m,j}} (1 + T^{p^{j+1}})^{l_{j+1} t_1 + \cdots + l_{j+1} t_{j} + l_1 t_{j} + l_0 t_{j+1} + a_{m,j+1}} \cdots \pmod{p},$$

(3.7)

where $a_{m,j} \geq 0$ for all $j \geq 2$.

Furthermore, for $j \geq 0$,

$$a_{l_k + j, p^j + l_{k-1} p^j + \cdots + l_1 p + l_0, k+1} = a_{l_{k-1} p^{k-1} + \cdots + l_1 p + l_0, k+1}$$

(3.8)

**Proof.** One can easily deduce (3.7) from (3.6). We now give a proof of (3.8). Let $m_1 = l_k p^k + \cdots + l_1 p + l_0$. From (3.7), the exponent of $(1 + T^{p^{j+1}})$ in the expansion of $(1 + T)^{m_1}$ and $(1 + T)^u^{l_k + l_{k-1} p^j + \cdots + l_1 p + l_0}$ are, respectively

$$l_{k-1} t_1 + \cdots + l_1 t_1 + l_0 t_{k+1} + a_{m_1,k+1}$$

(3.9)

$$l_k t_1 + l_{k-1} t_2 + \cdots + l_1 t_k + l_0 t_{k+1} + a_{l_k + j, p^j + \cdots + l_1 p + l_0, k+1}.$$ 

(3.10)

Again,

$$u^{l_k + l_{k-1} p^j + \cdots + l_1 p + l_0} = 1 + l_k t_1 p^{k+1} + \cdots.$$ 

Hence,

$$(1 + T)^{u^{l_k + l_{k-1} p^j + \cdots + l_1 p + l_0}} \equiv (1 + T)(1 + T^{p^{k+1}})^{l_k t_1 + \cdots} \pmod{p}.$$ 

This implies that modulo $p$, the exponent of $(1 + T^{p^{k+1}})$ in the expansion of $(1 + T)^{u^{l_k + l_{k-1} p^j + \cdots + l_1 p + l_0}}$ is

$$l_k t_1 + l_{k-1} t_2 + \cdots + l_1 t_k + l_0 t_{k+1} + a_{m_1,k+1}.$$ 

(3.11)

From (3.10) and (3.11), we complete the proof of the lemma. \hspace{1cm} \Box

In [1], we proved the following two theorems.

**Theorem 3.4.** For non-negative integers $n, t, k$, we have

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{ti + k}{n} = t^n.$$ 

(3.12)
Theorem 3.5. For non-negative integers $n, t, k, j$ with $n \geq j \geq 1$, we have
\[
\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \binom{t_i + k}{n-j} = 0. \tag{3.13}
\]

We now state a result from [9].

Result 3.6. Suppose that $n \geq m$ satisfy
\[
\begin{align*}
  n &= a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k \\
  m &= b_0 + b_1 p + b_2 p^2 + \cdots + b_k p^k
\end{align*}
\]
where $a_j, b_j \in \{0, 1, \ldots, p - 1\}$. Then
\[
\binom{n}{m} \equiv \binom{a_0}{b_0} \binom{a_1}{b_1} \cdots \binom{a_k}{b_k} \pmod{p}.
\]

Lemma 3.7. Let $l, m \geq 0$. Suppose that $l = l_0 + l_1 p + \cdots + l_k p^k$ with $0 \leq l_i < p$ for $i = 1, 2, \ldots, k$. Then
\[
a_l(f_m) \equiv \begin{cases} 
  0 \pmod{p}, & \text{if } m < pl; \\
  t_{l_0+i_1 t_2+\cdots+i_k t^k} \pmod{p}, & \text{if } m = pl.
\end{cases}
\]

Proof. Since $m < pl$, we can write $m = m_0 + m_1 p + \cdots + m_{k+1} p^{k+1}$, where $0 \leq m_i < p$.

From (3.2), we have
\[
a_{l_0+i_1 t_2+\cdots+i_k t^k}(f_m) \\
= \sum_{j=0}^{l_0+i_1 t_2+\cdots+i_k t^k-j} (-1)^{l_0+i_1 t_2+\cdots+i_k t^k-j} \binom{l_0 + l_1 p + \cdots + l_k p^k}{j} \binom{u^j}{m}.
\tag{3.14}
\]

But, $\binom{u^j}{m}$ is the coefficient of $T^m$ in the expansion of $(1 + T)^{u^j}$. Clearly from (3.7), the coefficient of $T^m$ modulo $p$ in $(1 + T)^{u^j}$ is zero if $1 < m_0 < p$. Also, the coefficients of $T^m$ modulo $p$ in $(1 + T)^{u^j}$ are equal for $m_0 = 0, 1$. So, we can assume that $m_0 = 0$.

Letting $j = i_k p^k + i_{k-1} p^{k-1} + \cdots + i_1 p + i_0$ and using (3.7) and then (3.8), we find that modulo $p$,
\[
\binom{u^j}{m} \equiv \binom{i_0 t_1}{m_1} \binom{i_1 t_2 + a_{j_2}}{m_2} \cdots \binom{i_k t_1 + i_{k-1} t_2 + \cdots + i_1 t_k + i_0 t_{k+1} + a_{j,k+1}}{m_{k+1}}
\equiv \binom{i_0 t_1}{m_1} \binom{i_1 t_2 + a_{i_2}}{m_2} \cdots \binom{i_k t_1 + + \cdots + i_1 t_k + i_0 t_{k+1} + a_{i_k p^{k-1}+\cdots+i_{0,k+1}}}{m_{k+1}}
\]
We now use result 2.6 to simplify (3.14) and deduce that
\[
a_{l_0+I_1p+\ldots+l_kp^k}(f_{m_1p+\ldots+m_{k+1}p^{k+1}}) \\
\equiv \sum_{i_0=0}^l (-1)^{l_0-i_0} \binom{l_0}{i_0} \binom{i_0t_1}{m_1} \times \left[ \sum_{i_1=0}^l (-1)^{l_1-i_1} \binom{l_1}{i_1} \binom{i_1t_1 + a_{i_0,2} + i_0t_2}{m_2} \right] \\
\times \left[ \sum_{i_2=0}^l (-1)^{l_2-i_2} \binom{l_2}{i_2} \binom{i_2t_1 + i_1t_2 + a_{i_2p+i_o,3} + i_0t_3}{m_3} \right] \times \ldots \times \left[ \sum_{i_{l-1}=0}^l (-1)^{l_{l-1}-i_{l-1}} \binom{l_{l-1}}{i_{l-1}} \binom{i_{l-1}t_1 + \cdots + i_1t_{l-1} + a_{i_{l-2}p^{l-2}+\ldots+i_1p+i_0,j + i_0t_j}{m_j} \right].
\] (3.15)

If \( m < pl \), then there exists \( j \) such that \( m_j < l_{j-1} \) with \( j \in \{0,1,\ldots,k+1\} \) and \( m_i = l_{i-1} \) for \( i > j \). Using (3.12) in (3.15), we obtain
\[
a_{l_0+I_1p+\ldots+l_kp^k}(f_{m_1p+\ldots+m_{k+1}p^{k+1}}) \\
\equiv t_1^{l_1+\ldots+l_j} \left[ \sum_{i_0=0}^l (-1)^{l_0-i_0} \binom{l_0}{i_0} \binom{i_0t_1}{m_1} \times \left[ \sum_{i_1=0}^l (-1)^{l_1-i_1} \binom{l_1}{i_1} \binom{i_1t_1 + a_{i_0,2} + i_0t_2}{m_2} \right] \right] \\
\times \left[ \sum_{i_2=0}^l (-1)^{l_2-i_2} \binom{l_2}{i_2} \binom{i_2t_1 + i_1t_2 + a_{i_2p+i_o,3} + i_0t_3}{m_3} \right] \times \ldots \times \left[ \sum_{i_{l-1}=0}^l (-1)^{l_{l-1}-i_{l-1}} \binom{l_{l-1}}{i_{l-1}} \binom{i_{l-1}t_1 + \cdots + i_1t_{l-1} + a_{i_{l-2}p^{l-2}+\ldots+i_1p+i_0,j + i_0t_j}{m_j} \right].
\] (3.16)

But, \( m_j = l_{j-1} - (l_{j-1} - m_j) \) with \( l_{j-1} - m_j > 0 \). Using (3.13), we have
\[
a_{l_0+I_1p+\ldots+l_kp^k}(f_{m_1p+\ldots+m_{k+1}p^{k+1}}) \equiv 0 \pmod{p}
\]
If \( m = pl \), then \( m_i = l_{i-1} \) for all \( i = 1,\ldots,k+1 \). Hence from (3.12), we have
\[
a_{l_0+I_1p+\ldots+l_kp^k}(f_{l_0p+l_1p^2+\ldots+l_kp^{k+1}}) \equiv t_1^{l_0+I_1p+\ldots+l_k} \pmod{p}
\]
This completes the proof of the theorem. \( \square \)

**Lemma 3.8.** Let \( j_i, m_i \geq 0 \) for \( i = 1,\ldots,n \). Then
\[
a_{j_1,\ldots,j_n}(f_{m_1,\ldots,m_n}) \equiv \begin{cases} 0 \pmod{p}, & \text{if } m_i < pj_i \text{ for some } i; \\
a \text{-adic unit } \pmod{p}, & \text{if } m_i = pj_i \text{ for all } i. \end{cases}
\]

**Proof.** The proof follows from the lemma 3.7 and the fact that
\[
a_{j_1,\ldots,j_n}(f_{m_1,\ldots,m_n}) = \prod_{i=1}^n a_{j_i}(f_{m_i}).
\]

\( \square \)

**Proof of Theorem 2.1:** Recall that
\[
\hat{\beta}(T_1,\ldots,T_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} b_{m_1,\ldots,m_n}(T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}
\]
\[ \widehat{\beta}(T_1, \ldots, T_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} g_{m_1, \ldots, m_n} (T_1 - 1)^{m_1} \cdots (T_n - 1)^{m_n}. \]

We know that \( \mu(\Gamma_0) = \mu(\beta) \), that is, \( \mu(\widehat{\beta}(T_1, \ldots, T_n)) = \mu(\beta) \). For any power series \( F(T_1, \ldots, T_n) \in \mathcal{O}[[T_1 - 1, \ldots, T_n - 1]] \), if \( \pi|F(T_1, \ldots, T_n) \) then \( \lambda(\pi^{-1}F(T_1, \ldots, T_n)) = \lambda(F(T_1, \ldots, T_n)) \). So, we may assume that \( \mu(\widehat{\beta}(T_1, \ldots, T_n)) = 0 \).

Suppose that \( \lambda(\Gamma_0) = k \), that is, \( \lambda(\widehat{\beta}(T_1, \ldots, T_n)) = k \). If \( k = 0 \), then \( g_{0, \ldots, 0} \) and \( b_{0, \ldots, 0} \) are units in \( \mathcal{O} \) and hence \( \lambda(\Gamma_0) = 0 = p\lambda(\beta) \). If \( k \geq 1 \), then there exists a partition \( k_1 + \cdots + k_n \) of \( k \) such that \( g_{k_1, \ldots, k_n} \) is a unit in \( \mathcal{O} \) and for every \( m_i \geq 0 \) satisfying \( m_1 + \cdots + m_n < k \), \( g_{m_1, \ldots, m_n} \equiv 0 \pmod{\pi} \). Let \( r < pk \). Let \( r = r_1 + \cdots + r_n \) and \( k = i_1 + \cdots + i_n \) be any partitions of \( r \) and \( k \), respectively. If \( l_i = \text{ord}_p(r_i!) \), then from (3.5) we get

\[
b_{r_1, \ldots, r_n} \equiv \sum_{j_1=0}^{l_1} \cdots \sum_{j_n=0}^{l_n} g_{j_1, \ldots, j_n} a_{j_1, \ldots, j_n} (f_{r_1, \ldots, r_n}) \pmod{\pi}. \tag{3.17} \]

If \( j_1 + \cdots + j_n \geq k \), then \( p j_1 + \cdots + p j_n \geq pk > r \). Hence \( r_i < p j_i \) for some \( i \) and lemma 3.8 implies that

\[
a_{j_1, \ldots, j_n} (f_{r_1, \ldots, r_n}) \equiv 0 \pmod{\pi}. \tag{3.18} \]

Again if \( j_1 + \cdots + j_n < k \), then \( g_{j_1, \ldots, j_n} \equiv 0 \pmod{\pi} \). Thus if \( r < pk \), then (3.18) and (3.17) imply that

\[
b_{r_1, \ldots, r_n} \equiv 0 \pmod{\pi} \tag{3.19} \]

for every partition \( r = r_1 + \cdots + r_n \).

Now let \( r = pk \). Consider the partition \( k_1 + \cdots + k_n \) of \( k \). Then \( pk_1 + \cdots + pk_n \) is a partition of \( pk \) such that \( \text{ord}_p((pk_i)!)) = \text{ord}_p(r_i!) = l_i \geq k_i \). From (3.5) and lemma 3.8, we find that

\[
b_{pk_1, \ldots, pk_n} \equiv \sum_{j_1=0}^{l_1} \cdots \sum_{j_n=0}^{l_n} g_{j_1, \ldots, j_n} a_{j_1, \ldots, j_n} (f_{pk_1, \ldots, pk_n}) \]
\[
\equiv g_{k_1, \ldots, k_n} a_{k_1, \ldots, k_n} (f_{pk_1, \ldots, pk_n}) \pmod{\pi}, \tag{3.20} \]

which is a unit in \( \mathcal{O} \). This proves that \( \lambda(\beta) = pk_1 + \cdots + pk_n = pk = p\lambda(\widehat{\beta}(T_1, \ldots, T_n)) \). This completes the proof of the main theorem. \qed

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Iwasawa $\lambda$-invariants of $p$-adic measures on $\mathbb{Z}_p^n$ and their $\Gamma$-transforms

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