Main conjecture for totally real fields

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- $\mathcal{G} := Gal(F_\infty/F)$ $\quad H := Gal(F_\infty/F^{cyc})$
  $\Gamma := Gal(F^{cyc}/F) \cong \mathbb{Z}_p$ and $X := Gal(M/F_\infty)$.

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- Extend this action to the action of $\Lambda(\mathcal{G}) := \mathbb{Z}_p[[\mathcal{G}]] := \varprojlim_U \mathbb{Z}_p[\mathcal{G}/U]$.
- We wish to study X as a $\Lambda(\mathcal{G})$-module.

- Let $C^{\cdot}(F_\infty/F) = RHom(R\Gamma_{et}(O_\infty([\frac{1}{\Sigma}]), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p)$.

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- We assume that there is an open pro-p subgroup J of $\mathcal{G}$ such that the cyclotomic μ invariant of F_∞^J vanishes. This is known to be true if F_∞^J is abelian over $\mathbb{Q}$ (Ferrero-Washington).

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- The assumtion implies that the cohomologies of $C^{\cdot}(F_\infty/F)$ are S-torsion.

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- here $K_0(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})_S)$ is the Grothendieck group of the category of bounded complexes of finitely generated projective $\Lambda(\mathcal{G})$ with S-torsion cohomologies.
- A characteristic element of $C^{\cdot}(F_\infty/F)$ is any element $f \in K_1'(\Lambda(\mathcal{G})_S)$ such that $\partial(f) = [C^{\cdot}(F_\infty/F)]$.

Theorem

(Main Conjecture) There is a unique element $\zeta = \zeta(F_\infty/F) \in K_1'(\Lambda(\mathcal{G})_S)$ *such that*
(i) $\partial(\zeta) = -[C^{\cdot}(F_\infty/F)]$, *and*
(ii) For any Artin representation ρ *of* $\mathcal{G}$ *and any positive integer r divisible by* $p-1$, *we have*

$$\zeta(\rho\kappa_F^r) = L_\Sigma(\rho, 1-r).$$

Here $\kappa_F : Gal(F(\mu_{p^\infty})/F) \to \mathbb{Z}_p^\times$ *is the p-adic cyclotomic character.*

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- When $\mathcal{G}$ is abelian and H is finite, the theorem is an easy consequence of the classical Iwasawa main conjecture as proven by Wiles.
- The above theorem is also a consequence of the series of paper by Ritter-Weiss and a recent paper of Burns.

Reduction to one dimensional case

Theorem

The main conjecture is true for F_∞/F if and only if it is true for F_∞^U/F for every open subgroup U of H which is normal in $\mathcal{G}$.

Reduction to hyperelementary groups

- A finite group P is called l-hyperelementary if there is an l-group π and a finite cyclic group C_n of order n such that l does not divide n and $P \cong C_n \rtimes \pi$. A finite group P is hyperelementary if it is l-hyperelementary for some l.

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- $\mathcal{G} \cong H \rtimes \Gamma$. Let Γ^{p^e} be a fixed central subgroup of $\mathcal{G}$ and let $G := \mathcal{G}/\Gamma^{p^e}$. For any subgroup P of G, let U_P be the inverse image of P in $\mathcal{G}$.

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- Theorem

  The main conjecture for one dimensional p-adic Lie extensions F_∞/F is true if and only if for every hyperelementary subgroup P of G, the main conjecture is true for $F_\infty/F_\infty^{U_P}$.

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- The case when G is p-hyperelementary is reduced to the case when G is a p group i.e. when $\mathcal{G}$ is a one dimensional pro-p p-adic Lie group.

- Now on we assume that F_∞/F is a p-adic Lie extension of dimension 1 and $\mathcal{G} = Gal(F_\infty/F)$ is a pro-p group. Then $\mathcal{G} = H \rtimes \Gamma$.

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- and let $T_{P,S}$ be the image of $\Lambda(U_P)_S$ under this map.

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- For $P \leq P'$, two cyclic subgroups of G, we let $ver_P^{P'}$ denote the map

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- Theorem

The main conjecture for F_∞/F is true if and only if the following congruence holds: for all cyclic subgroups P of G

$$\zeta_P \equiv \sum_{P'} ver_P^{P'}(\zeta_{P'})(mod\ T_{P,S}),$$

where the sum ranges over all cyclic subgroup P' of G such that $P'^p = P$ and $P' \neq P$.