Vector Bundles and p-adic Hodge Theory

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Introduction

$F=$ algebraically closed field complete with respect to a non trivial absolute value.
$B_? =$ the completion of a reasonably choosen subring for a reasonably choosen topology.
If $F = \mathbb{C}$, we have the Weierstrass preparation theorem.
If F is not archimedean, there is an analogue of Weierstrass preparation theorem (Hensel, Lazard,...)

In this talk, I want to
1 – Redo Lazard's work and extends it to the mixed char. case.
2 – If K is a p-adic field, give a rough idea of how one can redo (and extend) p-adic Hodge theory by viewing a p-adic representation of $G_K = \mathrm{Gal}(\overline{\mathrm{K}}/\mathrm{K})$ as a special kind of G_K-equivariant vector bundles over a suitable ≪ curve ≫ X.

The ring A^b

For simplicity we assume $F=$ algebraically closed field complete with respect to a non trivial non archimedean absolute value $|\ |$,

$$\mathcal{O}_F = \left\{ a \in F \mid |a| \leq 1 \right\} \ , \ \mathfrak{m}_F = \left\{ a \in F \mid |a| < 1 \right\} \text{ and}$$

– (EC) either $A^b = \mathcal{O}_F[[\pi]]$ (equal char. case),
– (MC) $\mathrm{char}(F) = p$ and $A^b = W(\mathcal{O}_F)$ (mixed char. case).

In (EC), set $[a] = a$ for all $a \in \mathcal{O}_F$.
In (MC) set $[a] = (a, 0, 0, \ldots, 0, \ldots)$ for all $a \in \mathcal{O}_F$ and $\pi = p$.
Any element of A^b can be uniquely written

$$f = \sum_{n=0}^{\infty} [a_n] \pi^n \quad \text{with } a_n \in \mathcal{O}_F \ .$$

Let $a_0 \in \mathfrak{m}_F$, non zero and $I_0 = ([a_0], \pi)$. Then A^b is a local ring, $\mathfrak{m}_{A^b} = \sqrt{I_0}$ and A^b is separated and complete for the I_0-topology.

Some prime ideals of A^b (1)

$k_F = \mathcal{O}_F/\mathfrak{m}_F$.

Proposition

The ideal $\mathfrak{p}_0$ of A^b generated by the $[a]$ for $a \in \mathfrak{m}_F$ is a prime ideal. Its closure $\widehat{\mathfrak{p}}_0$, also a prime ideal, is

$$\widehat{\mathfrak{p}}_0 = \left\{ \sum_{n=0}^{+\infty} [a_n]\pi^n \in A^b \mid a_n \in \mathfrak{m}_F,\ \forall n \right\} ,$$

kernel of the projection of A^b onto $k_F[[\pi]]$ (in the (EC) case) (resp. $W(k_F)$ in the (MC) case).

If $f \in A^b$, we denote $\overline{f}$ its image in $k_F([[\pi]]$ (resp. $W(k_F)$) and set

$$\deg(f) = v(\overline{f})\ (\in \mathbb{N} \cup \{+\infty\}) .$$

$$\deg(f) = +\infty \iff f \in \widehat{\mathfrak{p}}_0 ,$$
$$\deg(f) = 0 \iff f \text{ is invertible in } A^b .$$

The map θ

Let C be an algebraically closed field of char. 0, complete with respect to a non trivial, non archimedean absolute value, whose residue field is of characteristic $p > 0$.

$$F = F(C) = \left\{ x = (x^{(n)})_{n \in \mathbb{N}} \mid x^{(n)} \in C \ , \ (x^{(n+1)})^p = x^{(n)} \right\} ,$$

with $(x+y)^{(n)} = \lim_{m \mapsto +\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$, $(xy)^{(n)} = x^{(n)} y^{(n)}$, is an algebraically closed field of characteristic p, complete for the absolute value

$$|x| = |x^{(0)}| \ .$$

The map $\theta : W(\mathcal{O}_F) \to \mathcal{O}_C$, defined by

$$\theta((b_0, b_1, \ldots, b_n, \ldots)) = \sum_{n=0}^{+\infty} p^n b_n^{(n)} \ ,$$

is a surjective homomorphism of rings whose kernel is a principal ideal $\mathfrak{p}_C$. If $\lambda \in \mathfrak{m}_F$ is such that $\lambda^{(0)} = p$, then $p - [\lambda]$ is a generator.

Some prime ideals of A^b (2)

Proposition

i) If $f \in A^b$ is of degree 1, the ideal generated by f is prime.
ii) Let $\mathfrak{p}$ a non zero prime ideal of A^b of finite type. There exists $\lambda \in \mathfrak{m}_F$ such that $\mathfrak{p}$ is the principal ideal generated by $\pi - [\lambda]$ (of degree 1).
iii) If $A^b = W(\mathcal{O}_F)$ and if $\mathfrak{p} = (p - [\lambda])$ with $\lambda \neq 0$, then $A^b/\mathfrak{p}$ is the unit ball $\mathcal{O}_{C_\mathfrak{p}}$ of a field $C_\mathfrak{p}$ of characteristic 0, algebraically closed and complete for a non trivial non archimedean absolute value, whose residue field is of characteristic p.
The map $\mathcal{O}_F \to \mathcal{O}_{F(C_\mathfrak{p})}$ sending a to $(\theta_\mathfrak{p}([a^{p-n}]))_{n\in\mathbb{N}}$ is an isomorphism. Using it to identify these two rings, the projection

$$A^b = W(\mathcal{O}_F) \to \mathcal{O}_{C_\mathfrak{p}}$$

is the map θ defined on the previous slide.

The closed points of X

If F is of characteristic p, A^b is equipped with the Frobenius automorphism defined by

$$\varphi\Big(\sum_{n=0}^{+\infty}[a_n]\pi^n\Big) = \sum_{n=0}^{+\infty}[a_n^p]\pi^n \ .$$

Y = set of principal prime ideals of A^b, different from $\{0\}$ and (π). The cyclic group generated by φ acts on Y. We'll define later, for each integer $h \geq 1$ a scheme X_h, whose closed points will be the quotient $Y/\varphi^{h\mathbb{Z}}$. This will be a cyclic covering of degree h of $X = X_1$.

Completions

Fix $q > 1$. For $0 \leq \rho \leq 1$, we have an absolute value on B^b defined by :

$$|\sum_{n>>-\infty} [a_n]\pi^n|_\rho = \begin{cases} \sup_{n\in\mathbb{Z}}\{|a_n|\rho^n\} & \text{if } \rho \neq 0 \ , \\ q^{-m} \text{ if } a_n = 0 \text{ for } n < m \text{ and } a_m \neq 0 & \text{if } \rho = 0 \ . \end{cases}$$

For any interval $I \subset [0, 1[$, let B_I (resp. B_I^+) be the completion of B^b (resp. $B^{b,+}$) for the family $(|\ |_\rho)_{\rho\in I}$.

Remark : Let $I \subset [0, 1[$ and $(a_n)_{n\in\mathbb{Z}}$ in F (resp. $\mathcal{O}_F$) such that $|a_n|\rho^n$ goes to 0 when $n \mapsto +\infty$ and when $n \mapsto -\infty$ for all $\rho \in I$. Then

$$\sum_{n\in\mathbb{Z}} [a_n]\pi^n \in B_I \text{ (resp. } B_I^+) \ .$$

If (EC), any element of B_I (resp. B_I^+) may be written uniquely like that. Doesn't seem to be always true in mixed char..

Special cases

We have natural injective maps $B_I^+ \subset B_I$, $B_J^+ \subset B_I^+$, $B_J \subset B_I$ (if $I \subset J$). Also $B+_I = B^+_{\widetilde{I}^1}$ if $\widetilde{I}^1$ is the smallest interval containing I whose closure contains 1.

If $\mathrm{char}(F) = p$, φ extends to $B^{b,+}$, to B^b, to isomorphisms

$$B_I \to B_{\varphi(I)} \text{ and } B_I^+ \to B^+_{\varphi(I)} \text{ where } \varphi(I) = \{\rho^p \mid \rho \in I\} ,$$

and to automorphisms of the rings

$$B^+ = B^+_{]0,1[} \ , \ B^- = \varinjlim_{\rho>0} B_{[0,\rho]} \text{ and } B = \varinjlim_{\rho>0} B_{]0,\rho]} \ .$$

Get

$$B = B^+ + B^- \text{ and } B^{b,+} = B^+ \cap B^- \ .$$

Factorisation (1)

(Y= set of prime principal ideals of A^b different from $\{0\}$ and (π))

If $\mathfrak{p} = (\pi - [\lambda]) \in Y$, set $|\mathfrak{p}| = |\lambda|$ (independent of the choice of λ).
If $I \subset [0, 1[$ is an interval, set $Y_I = \{\mathfrak{p} \in Y \mid |\mathfrak{p}| \in I\}$, then

$$B_I/(B_I\mathfrak{p}) = C_\mathfrak{p} \text{ if } \mathfrak{p} \in Y_I \text{ and } B_I\mathfrak{p} = B_I \text{ if } \mathfrak{p} \notin Y_I \ .$$

Same is true for B_I^+ if 1 belongs to the closure of I.
If I is closed, B_I is a principal domain whose maximal ideals are the $B_I\mathfrak{p}$ for $\mathfrak{p} \in Y_I$.
In general, if $f \in B_I$ is non zero, set, for any $\mathfrak{p} \in Y_I$,

$$v_\mathfrak{p}(f) = \text{ biggest integer } r \text{ such that } f \in (B_I\mathfrak{p})^r \ .$$

If $\rho_1, \rho_2 \in I$ are such that $0 < \rho_1 \leq \rho_2$, we have $v_\mathfrak{p}(f) = 0$ for almost all $\mathfrak{p}$ satisfying $\rho_1 \leq |\mathfrak{p}| \leq \rho_2$.
For each $\mathfrak{p} \in Y_I$, choose $\lambda_\mathfrak{p} \in \mathcal{O}_F$ such that $p - [\lambda_\mathfrak{p}]$ generates $\mathfrak{p}$.

Factorisation (2)

Weierstrass preparation theorem

Let I be an interval contained in $[0,1[$ and $\widetilde{I}^0$ the smallest interval containing I and 0. If $f \in B_I$ is non zero and if $\rho \in I$, then $f = f_{\leq\rho}.f_{>\rho}$ with

$$f_{\leq\rho} = \prod_{|\mathfrak{p}|\leq\rho} \left(1 - \frac{[\lambda_\mathfrak{p}]}{\pi}\right)^{v_\mathfrak{p}(f)}$$

and $f_{>\rho} \in B_{\widetilde{I}^0}$, invertible in $B_{[0,\rho]}$.
Moreover

$$f \in B_I^+ \iff f_{>\rho} \in B^{b,+} \ .$$

The schemes X_h

$\mathrm{char}(F) = p$, $h \geq 1$. We set

$$X_h = \mathrm{Proj}(P_h). \text{ and } X := X_1 \ ,$$

where P_h is the graded algebra

$$P_h = \bigoplus_{d \geq 0} \underbrace{(B^+)^{\varphi^h = \pi^d}}_{P_{h,d}}$$

$P_{h,0} = \mathbb{F}_{p^h}((\pi))$ in (EC) case (resp. $\mathbb{Q}_{p^h}$ in (MC) case)..
Each $P_{h,d}$ is a Banach space over $P_{h,0}$.
P_h is generated, as a $\mathbb{Q}_{p^h}$-algebra, by $P_{h,1}$.

Factorisation in P_h

Proposition

Let $h \geq 1$ and $x \in Y/\varphi^{h\mathbb{Z}}$. Choose $\mathfrak{p} = (\pi - [\lambda]) \in x$. The set V_x of the $t \in P_{h,1}$ such that $t \in \mathfrak{p}$ is a one dimensional sub-$P_{h,0}$-vector space of $P_{h,1}$ independent of the choice of $\mathfrak{p}$.

Proposition

Let $d \geq 2$ and let $u \in P_{h,d}$ non zero. Then

$$u = t_1 t_2 \ldots t_d$$

with the t_i's in $P_{h,1}$, uniquely determined up to permutation and multiplication by a non zero element of $P_{h,0}$.

The curves X_h

$h \geq 1$, $X_h = \mathrm{Proj}(P_h)$, $X := X_1$.

Theorem

1. *X_h is a regular noetherian scheme of dimension* 1
2. $X_h \simeq X \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^h}$ *via* $\underbrace{(B^+)^{\varphi=\pi^d}}_{P_{1,d}} \hookrightarrow \underbrace{(B^+)^{\varphi^h=\pi^{hd}}}_{P_{h,hd}}$
3. *Choose $t \in P_{1,1} \setminus \{0\}$, then $P_1 t$ is an homogeneous prime ideal of P_1 defining a closed point ∞ of X :*
   - 3.1 *if $\mathfrak{p} \in Y$ contains t, the residue field at ∞ is the fraction field of $A^b/\mathfrak{p}$.*
   - 3.2 *$X \setminus \{\infty\} = Spec(B_e)$ with $B_e = \{x \in B^+[\frac{1}{t}] \mid \varphi(x) = x\}$ and B_e is a principal domain !*
4. *The map $t \mapsto \infty$ induces a bijection between the $\mathbb{F}_p((\pi))$ (resp. $\mathbb{Q}_p$)-lines in $P_{1,1}$ and the set $|X|$ of closed points of X.*

Line bundles

$h \geq 1$, $d \in \mathbb{Z}$

$$\mathcal{O}_{X_h}(d) = \widetilde{P_h[d]} \text{ is a line bundle on } X_h \ .$$

$$P_h = \bigoplus_{d \in \mathbb{Z}} H^0(X_h, \mathcal{O}_{X_h}(d))$$

Theorem

1. $d \mapsto [\mathcal{O}_X(d)]$ *induces an isomorphism* $\mathbb{Z} \underset{\simeq}{\overset{\deg}{\rightleftarrows}} Pic(X)$.
2. $H^1(X, \mathcal{O}_X) = 0$ *(« genus 0 » like* $\mathbb{P}^1$*).*
3. $H^1(X, \mathcal{O}_X(-1)) \neq 0$ *(unlike* $\mathbb{P}^1$*)*
   (⇒ there exists a non split short exact sequence
   $0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X(1) \to 0$ *.)*

Vector bundles (1)

Any non zero vector bundle $\mathcal{E}$ over X has a rank $r(\mathcal{E}) \in N^*$, a degree $d(\mathcal{E}) \in \mathbb{Z}$ and a slope $\mu(\mathcal{E}) = d(E)/r(E) \in \mathbb{Q}$.
A vector bundle $\mathcal{E}$ is semi-stable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for any sub vector bundle $\mathcal{F}$.
The Harder-Narasimhan theorem holds : If $\mathcal{E}$ is a vector bundle, there is a unique filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_{i-1} \subset \mathcal{E}_i \subset \ldots \subset \mathcal{E}_{m-1} \subset \mathcal{E}_m = \mathcal{E}$$

by strict sub vector bundles such that each $\mathcal{E}_i i/\mathcal{E}_{i-1}$ is non zero and semi-stable and that

$$\mu(\mathcal{E}_1/\mathcal{E}_0) > \mu(\mathcal{E}_2/\mathcal{E}_1) > \ldots > \mu(\mathcal{E}_m/\mathcal{E}_{m-1}) .$$

Moreover, *the Harder-Narasiman filtration splits* (non canonically).

For $\lambda \in \mathbb{Q}$, set $\lambda = d/h$ with $h \geq 1$ and $(d,h) = 1$. Set $\mathcal{O}_X(\lambda) = \pi_{h*}\mathcal{O}_{X_h}(d)$. Then

Vector bundles (2)

Theorem

Let $\lambda \in \mathbb{Q}$. Then

1. *$\mathcal{O}_X(\lambda)$ is stable of slope λ.*
2. *A vector bundle is semi-stable of slope λ if and ony if $\mathcal{E}$ is isomorphic to a direct sum of copies of $O_X(\lambda)$.*

A stupid remark : *The functor*

$$\mathcal{E} \to H^0(X, \mathcal{E})$$

induces an equivalence of categories between semi-stable vector bundles over X of slope 0 and finite dimensonal $\mathbb{Q}_p$-vector spaces. The functor $V \mapsto V \otimes \mathcal{O}_X$ is a quasi-inverse.

An other remark :Let $\mathcal{E}$ a non zero vector bundle over X.Then,

– $H^0(X, \mathcal{E}) = 0$ if and only if all the Harder-Narasimhan slopes are < 0,

– The dimension over $\mathbb{Q}_p$ of $H^0(X, \mathcal{E})$ is finite if and only if all the Harder-Narasimhan slopes are ≤ 0,

– If $\dim_{\mathbb{Q}_p} H^0(X, \mathcal{E})$ is finite, we have $\dim_{\mathbb{Q}_p} H^0(X, \mathcal{E}) \leq r(\mathcal{E})$ with equality if and only if $\mathcal{E}$ is semi-stable of slope 0.

The arithmetic case

K= field of characteristic 0, complete with respect to a discrete valuation, with perfect residue field of characteristic $p > 0$.
$\overline{K}$= an algebraic closure of K, $C = p$-adic completion of $\overline{K}$, $F = F(C)$, $A^b = W(\mathcal{O}_F)$. Then

1. $B^+ = \widetilde{B}^+_{rig} = \cap_{n\in\mathbb{N}}\varphi^n(B^+_{cris})$,
2. $B^- = \widetilde{\mathcal{E}}^\dagger$, field of overconvergent functions,
3. $B = \widetilde{R}$ = Robba's ring,
4. With θ and t as above, $B_e = \{x \in B_{cris} \mid \varphi x = x\}$ (where $B_{cris} = B^+_{cris}[\frac{1}{t}]$)
5. $B^+_{dR} = \widehat{\mathcal{O}_{X,\infty}}$ (and $B_{dR} = \text{Frac } B^+_{dR} = B^+_{dR}[\frac{1}{t}]$).

$G_K = \mathrm{Gal}(\overline{K}/K)$ acts on C, B^+, X and fixes ∞ and $X^0 = X \setminus \{\infty\} = \text{Spec}(B_e)$. (therefore G_K acts on B_e, $\mathcal{O}_{X,\infty}$, B^+_{dR}, B_{dR},...)

We may consider G_K-equivariant bundles over X. From the stupid remark, we get an equivalence of categories

p-adic representations of G_K $\iff$ semi-stable G_K-equiv. bundles of slope 0 .

G_K-equivariant bundles over X

Let $\mathcal{E}$ be a vector bundle over X. We set

$$\mathcal{E}^0 = \Gamma(X \setminus \{\infty\}, \mathcal{E}) \ , \ \widehat{\mathcal{E}}^+_\infty = B^+_{dR} \otimes \mathcal{E}_\infty (\text{ completion of the fiber at } \infty \text{ })$$

$$\text{and } \iota : B_{dR} \otimes_{B_e} \mathcal{E} \simeq B_{dR} \otimes_{B^+_{dR}} \widehat{\mathcal{E}}^+_\infty \text{ the structural isomomorphism}$$

To know $\mathcal{E} \iff$ to know the triple

$$(\mathcal{E}^0, \widehat{\mathcal{E}}^+_\infty, \iota)$$

A G_K-equivariant bundle $\mathcal{E}$ over X may be identified to such a triple, with

1. $\mathcal{E}^O$ a B_e*-representation of* G_K, i.e. a free B_e-module of finite rank equipped with a semi-linear action of G_K,
2. $\widehat{\mathcal{E}}^+_\infty$ a B^+_{dR}*-representation of* G_K, i.e. a free B^+_{dR}-module of finite rank equipped with a semi-linear action of G_K,
3. $\iota : B_{dR} \otimes_{B_e} \mathcal{E} \to B_{dR} \otimes_{B^+_{dR}} \widehat{\mathcal{E}}^+_\infty$ a G_K-equivariant isomorphism of B_{dR}-vector spaces.

Classification of B_{dR}^+-representations which are B_{dR}-trivial

There is an obvious equivalence of categories

B_{dR}^+-representations of G_K which are B_{dR}-trivial	$\Longleftrightarrow$	finite dimensional filtered K-vector spaces

$$\widehat{\mathcal{E}}_\infty^+ \mapsto D_K = D_{dR}(\widehat{\mathcal{E}}_\infty^+) = (B_{dR} \otimes_{B_{dR}^+} \widehat{\mathcal{E}}_\infty^+)^{G_K} \text{ with } F^i D_K = (F^i B_{dR} \otimes_{B_{dR}^+} \widehat{\mathcal{E}}_\infty^+)^{G_K}$$

$$D_K \mapsto \widehat{\mathcal{E}}_\infty^+ = F^0(B_{dR} \otimes_K D_K) = \Sigma_{i \in \mathbb{Z}} F^{-i} B_{dR} \otimes_K F^i D_K \ .$$

The ring B_{log}

Set $B_{rig} = B^+\left[\frac{1}{t}\right] = \bigcap_{n\in\mathbb{N}} \varphi^n(B_{cris})$.

If $\mathfrak{m} = \text{Ker}\ (\theta : B^+ \to C) = (p - [\lambda])$, set

$$u = log\left(\frac{[\lambda]}{p}\right) = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{\left(\frac{[\lambda]}{p} - 1\right)^n}{n} \in B_{dR}^+ \ .$$

Set

$$B_{log} = B_{rig}[u] \subset B_{dR} \ .$$

u is transcendental over $\text{Frac}B_{rig}$ and $B_{log} = \bigcap_{n\in\mathbb{N}} \varphi^n(B_{st})$ (where $B_{st} = B_{cris}[u]$).

1. B_{log} is stable under G_K and $(B_{log})^{G_K} = K_0 = \text{Frac}(W(k))$ (where k is the residue field of K).
2. φ extends to B_{log} with $\varphi(u) = pu$.
3. $N = -\frac{d}{du} : B_{log} \to B_{log}$
4. $N\varphi = p\varphi N$.
5. If $g \in G_K$, $\varphi g = g\varphi$ and $Ng = gN$.

log-crystalline B_e-representations

A B_e-representation $\mathcal{E}^0$ is log-crystalline if the B_{log}-representation $B_{log} \otimes_{B_e} \mathcal{E}^0$ is trivial.

A (φ, N)-module over K_0 is a finite dimensional K_0-vector space D equipped with two operators $\varphi, N : D \to D$, with φ semi-linear with respect to the absolute Frobenius on K_0, N linear and $N\varphi = p\varphi N$.

Proposition

The category of log-crystalline B_e-representations is abelian and the functor

$$\begin{array}{ccc} \textit{log-crystalline } B_e\textit{-representations} & \to & (\varphi, N)\textit{modules over } K_0 \\ E^0 & \mapsto & D(\mathcal{E}^0) = (B_{log} \otimes_{B_e} \mathcal{E}^0)^{G_K} \end{array}$$

is an equivalence of categories. A quasi-inverse is given by the functor

$$D \quad \mapsto \quad (B_{log} \otimes_{K_0} D)_{N=0, \varphi=1}$$

It is a formal consequence of the fact that $(B_{log})_{N=0,\varphi=1} = B_e$.

log-crystalline G_K-bundles

Let $\mathcal{E}$ be a G_K-equivariant vector bundle over X which is log-crystalline (i.e. the B_e-representation $\mathcal{E}^0 = \Gamma(X \setminus \{\infty\}, \mathcal{E})$ is log-crystalline). The inclusion $B_{log} \subset B_{dR}$ implies that $\mathcal{E}$ is trivial at ∞. Moreover, if we set

$$D(\mathcal{E}) = D(\mathcal{E}^0) = (B_{log} \otimes_{B_e} \mathcal{E}^0)^{G_K} ,$$

then $D(\mathcal{E})$ has a natural structure of a *filtered (φ, N)-module over K*, i.e. this is a (φ, N)-module over K_0 and $D_K = K \otimes_{K_0} D(\mathcal{E})$ is a filtered K-vector space (because it may be identified to $D_{dR}(\widehat{\mathcal{E}}^0_\infty)$).

Theorem

The functor

$$\mathcal{E} \to D(\mathcal{E})$$

induces an equivalence of categories between log-crystalline G_K-bundles and filtered (φ, N)-modules over K.

This is a formal consequences of classifications of B_{dR}-trivial B_{dR}^+-repr's and of log-crystalline B_e-representations together with the fact that a vector bundle $\mathcal{E}$ over X is determined by the triple

$$(\mathcal{E}^0, \widehat{\mathcal{E}}^+_\infty, \iota)$$

where ι is the natural isomorphim $B_{dR} \otimes_{B_e} \mathcal{E}^0 \simeq B_{dR} \otimes_{B_{dR}^+} \widehat{\mathcal{E}}^+_\infty$.

Applications and complements

– The theorem ≪ weakly admissible implies admissible ≫ is a corollary of the previous theorem (We use the stupid remark and apply the theorem to the G_K-equivariant bundles which are semi-stable of slope 0).
– One can show that any G_K-equivariant vector bundle $\mathcal{E}$ which is trivial at ∞, or *de Rham* (i.e. such that $B_{dR} \otimes \mathcal{E}^0$ is trivial) is *potentially log-crystalline* (i.e. becomes log-crystalline after replacing K by a suitable finite extension).
This is actually a result on B_e-representations : to any B_e representation which is de Rham, one can associate a G_K-equivariant vector bundle. The Harder-Narasimhan filtration reduces the proof by dévissage to an old result of Shankar Sen and to a computation in Galois cohomology.
– The theorem ≪ de Rham implies potentially log-crystalline ≫ is the semi-stable of slope 0 case of this result.
– If $\mathcal{E}$ is a G_K-equivariant bundle over X which is de Rham, there exists an integer $h \leq 1$ and a finite extension L of K contained in $\overline{K}$ such that $\pi_h^*\mathcal{E}$ is a successive extensions of G_L-equivariant line bundles over X_h (generalisation of the notion of trianguline representations).