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On the modulo p Satake isomorphism with weight
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This is a common work with Guy Henniart.

1 Let p be a prime number, k a finite field of characteristic p, F a local non archimedean field of residue field k.

Let $\underline{G}$ be a connected reductive group over F, the group $G = \underline{G}(F)$ of F-rational points of $\underline{G}$.

A representation of G on a vector space is called smooth when the stabilizer of any vector is open in G.

It is interesting to study the complex irreducible smooth representations of G because they are local components of automorphic representations of adelic groups. The local counterpart of the congruences between automorphic representations are irreducible smooth representations of G over fields of positive characteristic.

Let $K \subset G$ be an open compact subgroup and let $\mathcal{H}(G, K)$ be the convolution ring of $Z[K\backslash G/K]$ (the functions on the double coset space $K\backslash G/K$ with Z-values).

Let $\mathbb{C}$ be the field of complex numbers.

$W \to W^K$ gives a bijection from the isomorphism classes of complex smooth irreducible representations W of G with $W^K \neq 0$ onto the isomorphism classes of simple $\mathcal{H}(G, K) \otimes_Z \mathbb{C}$-modules.

When $\mathbb{C}$ is replaced by a field C of characteristic p, this is false. Still it is interesting to study the $\mathcal{H}(G, K, C) := \mathcal{H}(G, K) \otimes_Z C$-modules.

Let C a commutative ring, B, L, U be closed subgroups of G such that $B = U.L$ is the semi-direct product of U normal with L. When we have an Iwasawa decomposition $G = BK$ and $B \cap K = (L \cap K)(U \cap K)$, there is a natural map

$$S \quad : \quad \mathcal{H}(G, K, C) \quad \to \quad \mathcal{H}(L, L \cap K, C)$$

$$Sf(x) := \sum_{u \in U/U \cap K} f(xu) \quad .$$

One says that S is a Satake isomorphism when S is injective and one can determine its image.

From now on, B is a minimal parabolic subgroup of G with Levi decomposition $B = U.L$. We say that K is L-special, if K fixes a special vertex of the apartment associated to L in the Bruhat-Tits building of G. Let $W := N_G(L)/L$ be the finite Weyl relative group.

2 The complex case $C = \mathbb{C}$.

Satake in 1963 did not considered S but $S\delta_B^{1/2}$ where δ_B is the modulus of B. For a maximal compact subgroup $K \subset G$ satisfying axiomatic conditions which were known to be true when G is a classical group and K a natural maximal open compact subgroup, the group $L \cap K$ is the unique maximal open compact subgroup of L, the quotient $L/L \cap K$ is commutative free finitely generated, the algebra $\mathcal{H}(L, L \cap K, \mathbb{C})$ is naturally isomorphic to

$\mathbb{C}[L/L\cap K]$ and that W acts on $L/L\cap K$. Satake showed that $S\delta_B^{1/2}$ is injective of image $\mathbb{C}[L/L\cap K]^W$. He showed also that $\mathcal{H}(G,K,Z)\simeq Z[T_1 \ldots, T_n]$ when G is simple with no center.

Later, Bruhat and Tits showed that any L-special maximal open compact subgroup $\tilde{K}$ of G satifies the axioms used by Satake, and the article of Cartier in Corvallis became the classical reference for the Satake isomorphism.

The Satake isomorphism, when G is classical and unramified (quasi-split and split over an unramified extension) and K a natural maximal open compact subgroup (an L-hyperspecial group), is the starting point of the definition by Langlands in 1970, of the L-group of G. Langlands sees the Satake isomorphism as a bijection between the isomorphism classes of complex smooth irreducible representations W of G with $W^K\neq 0$ and the irreducible characters of $\mathbb{C}[L/L\cap K]^W$, and those are in bijection with certain semi-simple conjugacy classes in the L-group of G.

Then Langlands define the partial L-functions of automorphic representations using the L-group and the fact that the local components of an irreducible automorphic representation are - except for a finite number of places - irreducible smooth representations associated to semi-simple conjugacy classes in the local L-group.

In 2009, Haines and Rostami considered the connected part K of an L-special maximal open compact $\tilde{K}$ of G, called a maximal L-special parahoric subgroup of G. They show that $\Lambda := L/(L\cap K)$ is commutative finitely generated of torsion subgroup $\Lambda_{tor}=(L\cap\tilde{K})/(L\cap K)$. The quotient group $\tilde{\Lambda}:=\Lambda/\Lambda_{tor}$ is $L/(L\cap\tilde{K})$. They adjusted the proof of Cartier to show that $S\delta_B^{1/2}$ is injective of image $\mathbb{C}[L/L\cap K]^W$.

When G splits over an unramifed extension of F, then $K=\tilde{K}$.

An example of a compact torus $G=\tilde{K}\neq K$ (Pappas-Rapoport)

$p\neq 2$, E/F is a ramified quadratic extension with uniformizers $p_E^2=p_F$ and $y\to\overline{y}$ the non trivial F-automorphism of E. Let $\underline{G}$ be the F-torus Ker (Norm : $R_{E/F}\mathbb{G}_m\to\mathbb{G}_m$).

Then $G=\{y\in E^*\mid y\overline{y}=1\}$ is compact and $G=\tilde{K}$. By Hilbert's theorem 90, there exists $x\in O_E$ such that $y=x/\overline{x}$. Then $K=\{y=x/\overline{x}\mid \mathrm{val}_E(x)\text{ even}\}$.

And also the group of unitary similitudes of a hermitian vector space of even dimension ≥ 4 over E for other examples.

3 Let C be a field of characteristic p. We cannot keep δ_B and we loose the symmetry by the Weyl group. Let $K\subset\tilde{K}$ as in Haines-Rostami.

An element $x\in L$ is called anti-dominant if $x^{-n}(U\cap K)x^n$ does not blow up when n goes to ∞. Then the elements of $x(L\cap\tilde{K})$ are also antidominant. Let $L^-,\Lambda^-,\tilde{\Lambda}^-$, be the monoids of anti-dominant elements. The commutative monoids $\Lambda^-,\tilde{\Lambda}^-$ are finitely generated.

The monoid $\tilde{\Lambda}^-$ is a fundamental domain for the action of W on $\tilde{\Lambda}$ and every element of Λ_{tor} is fixed by W.

Proposition. S is injective of image the functions in $\mathcal{H}(L,L\cap K,C)=C[L/(L\cap K)]$ with support in Λ^-.

In particular, the C-algebra $\mathcal{H}(G,K,C)$ is commutative of finite type.

We can deduce from Haines Rostami that $\mathcal{H}(G,K,Z) \subset \mathcal{H}(G,K,\mathbb{C})$ is commutative hence $\mathcal{H}(G,K,C) = \mathcal{H}(G,K,\mathbb{Z}) \otimes_{\mathbb{Z}} C$ is commutative.

3 The proposition is the particular case $V = C$ of a more general theorem.

Let C be any field and let $K \subset G$ be any open compact subgroup. Let V be an absolutely irreducible smooth C-representation of K. Let $\mathcal{H}(G,K,V)$ be the convolution algebra of functions $f : G \to \mathrm{End}_C(V)$ supported on finitely many cosets of $K\backslash G/K$ and satisfying $f(kgk') = kf(g)k'$ for all $k, k' \in K, g \in G$.

For a smooth C-representation W of G, $\mathrm{Hom}_K(V,W)$ is a module for $\mathcal{H}(G,K,V)$.

When $C = \mathbb{C}$, then $W \to \mathrm{Hom}_K(V,W)$ gives a bijection between the isomorphism classes of complex smooth irreducible representations W of G with $\mathrm{Hom}_K(V,W) \neq 0$ and the isomorphism classes of simple $\mathcal{H}(G,K,V)$-modules.

We can say more when V is a type. When V is trivial on the unique maximal normal pro-p-subgroup K_+ of K, one says that V is of level 0. The algebras $\mathcal{H}(G,K,V)$ for the types of level 0 when $K \subset G$ is a parahoric subgroup have been computed by Morris in 1993.

From now on C is a field of characteristic p and $K \subset \tilde{K}$ as in Haines-Rostami. Then K_+ acts trivially on V. The quotient K/K_+ is a finite group of Lie type. This is the main reason to replace $\tilde{K}$ by the parahoric K.

By the theory of representations of finite groups of Lie type, $\dim_C V^{U \cap K} = 1$ and $L \cap K$ acts on $V^{U\cap K}$ by a character χ, called the highest weight.

Let L_χ be the normalizer of χ in L, equal to the subgroup of $x \in L$ such that $\chi(xkx^{-1}) = \chi(k)$ for all $k \in K \cap L$. We have $L \cap K \subset L_\chi$ and $(L \cap K)/\operatorname{Ker}\chi$ is commutative.

Theorem (Henniart V.) The map

$$S \quad : \quad \mathcal{H}(G,K,V) \quad \to \quad \mathcal{H}(L, L\cap K, V^{U\cap K})$$

$$Sf(x) := \sum_{u \in U/U\cap K} f(xu) \quad .$$

is injective of image the functions in $\mathcal{H}(L, L\cap K, V^{U\cap K})$ supported on the anti-dominant submonoid L_χ^- of L_χ.

This is a theorem of Barthel and Livne (1993) when $G = GL(2,F)$, of Florian Herzig (2008) when G is unramified, K hyperspecial, F of characteristic 0. The Satake isomorphism with weight V is the first step of the classification by Barthel-Livne and Herzig of the irreducible admissible smooth representations of $GL(n,F)$ over an algebraic closure of k . In the case of Barthel-Livne and of Herzig, $\tilde{K} = K$, L is commutative hence $L = L_\chi$. Hence $\mathcal{H}(G,K,V)$ is commutative isomorphic by S to the subalgebra of $\mathcal{H}(L, L\cap K, V^{U\cap K})$ supported on L^-.

On the positive side, $\tilde{K} = K$ and $\mathcal{H}(G,K,V)$ is commutative for all V when G is semisimple simply connected or splits on an unramified extension.
But we have examples where $K = \tilde{K}$ and $\mathcal{H}(G,K,V)$ is NOT commutative.

The commutator $(x,y) = xyx^{-1}y^{-1}$ of two elements of L belongs to $L \cap K$. Let L'_χ be the subgroup of $x \in L$ such that $(x, L_\chi) \subset \operatorname{Ker}\chi$.

Theorem. The center of $\mathcal{H}(G,K,V)$ is the inverse image by S of the functions in $\mathcal{H}(L, L\cap K, V^{U\cap K})$ supported on the anti-dominant elements in L'_χ.

$\mathcal{H}(G,K,V)$ is a finitely generated module over its center and the center is a C-algebra of finite type.

Corollary. $\mathcal{H}(G,K,V)$ is commutative if and only if $L_\chi / \operatorname{Ker}\chi$ is commutative.

4 Example of a non commutative Hecke algebra $\mathcal{H}(G,K,V)$.

When G is a non abelian group of order 8, the center K has order 2 and G/K is abelian non cyclic of order 4. Let χ be a non trivial character of K. We have $G_\chi = G$ and $(G,G) \neq \{1\} = \operatorname{Ker}\chi$ hence $\mathcal{H}(G,K,\chi)$ is not abelian.

Suppose $p = q = 3$. Let E/F be a ramified quadratic extension, let $y \to \overline{y}$ be the non trivial Galois automorphism of E/F, let D/F be a division algebra of center F, reduced degree 4 over F and containing E, let $d \to Nrd(d)$ the reduced norm of D/F. Let

$$G : \{(d,x,y) \ \in \ D^* \times E^* \times E^* \quad | \quad Nrd(d)x^2 \frac{y}{\overline{y}} \ = \ 1 \ \} \quad .$$

The kernel of $(d,x,y) \mapsto (x,y) : G \to E^* \times E^*$ is the kernel D^1 of the reduced norm. The group G is $\underline{G}(F)$ for a reductive connected F-group $\underline{G}$, extension of an F-torus $\underline{T}$ by a reductive connected F-group $\underline{G}_1$

$$1 \to \underline{G}^1 \to \underline{G} \to \underline{T} \to 1 \quad .$$

Let U_E, U_D the group of units of the integers of E, D. Then

$$K \ = \ \tilde{K} \ = \ G \cap (U_D \times U_E \times U_E) \quad .$$

We have $|k^*| = |k_E^*| = 2$ and $|k_D^*| = 80 = 16 \times 5$. We have

$$K/K_+ = \{(z,t,u) \ \in \ k_D^* \times k^* \times k^* \quad | \quad n(z)t^2 \ = \ 1 \ \}$$

where $n(z) = z^{1+3+9+27} = z^{40}$. Hence

$$K/K_+ = \{(z,t,u) \ \in \ k_D^* \times k^* \times k^* \quad | \quad z \text{ is a square } \} \ .$$

Lemma. The algebra $\mathcal{H}(G,K,\chi)$ is not commutative when χ is the character of K inflating the character

$$\epsilon(z,t,u) \ = \ z^5 \quad .$$

We have $G_\chi = G$. Let $(d,x,y) \in G$. Then $Nrd(d)x^2\frac{y}{\overline{y}} \ = \ 1$ hence the valuation v of d is even. The conjugation by d induces on k_D the map $z \to z^{3^v}$. Clearly $3^v - 1$ is divisible by 8. When $z \in k_D^*$ is a square then $(z^{3^v-1})^5 = 1$.

We have $(G,G) \not\subset \operatorname{Ker}\chi$. We take two elements of G of the form $g = (d,1,y)$, $h = (p_D^2, Nrd(p_D)^{-1}, 1)$, for $d \in U_D$ of reduced norm $Nrd(d) = -1$ hence of reduction z with $z^{40} = -1$, $y \in E^*$ with $y^2 \in F^*$ and p_D an uniformizer of D. The reduction of $p_D^2 d p_D^{-2} d^{-1}$ is z^8. Hence $\chi(g,h) = z^{40} = -1$.

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