Unitarity and Analyticity Constraints on $\pi$-$K$ Form Factors

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Work done in collaboration with
Gauhar Abbas, Irinel Caprini and I. Sentitemsu Imsong
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Digression on $D - \pi$ form factors

Conclusions

Based on the papers: Gauhar Abbas, BA, I. Caprini and I. Sentitemsu Imsong, Physical Review, D 82 (2010) 094018
(see also, references therein)
Related investigations

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The matrix element for $K_{l3}^+$ has the structure:

$$T = \frac{G_F}{\sqrt{2}} V_{us} \ell^\mu F^+(p', p)$$

$$\ell^\mu = \bar{u}(p_\nu) \gamma^\mu (1 - \gamma_5) v(p_1)$$

$$F^+(p', p)_\mu = \langle \pi^0(p') | \bar{s} \gamma_\mu u | K^+(p) \rangle = \frac{1}{\sqrt{2}} ((p' + p)_\mu f_+(t) + (p - p')_\mu f_-(t))$$
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$f_+(t)$, $t = (p' - p)^2$ is known as the vector form factor as it is the P-wave projection of the crossed channel matrix element $\langle 0|\bar{s}\gamma_\mu u|K^+\pi^0, in\rangle$. 
The scalar form factor

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f_0(t) = f_+(t) + \frac{t}{M_K^2 - M_\pi^2} f_-(t)
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Definitions continued

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\[ f_0(t) = f_+(0) \left( 1 + \lambda'_0 \frac{t}{M_\pi^2} + \frac{1}{2} \lambda''_0 \frac{t^2}{M_\pi^4} + \cdots \right), \]

\( \lambda'_0 = M_\pi^2 \langle r_{\pi K}^2 \rangle / 6 \), \( \lambda''_0 = 2M_\pi^4 c \) are related to the radius \( \langle r_{\pi K}^2 \rangle \) and curvature, \( c \) used alternatively in the literature.
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Solutions of Muskelishvili-Omnès equations for form factors using phase shift information and some additional inputs to self-consistently generate them. Work of Moussallam, group of Jamin, Oller, Pich, Boito, Escribano.
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- Recent determinations from the lattice, e.g., RBC+UKQCD collaboration [P. A. Boyle et al., Physical Review Letters 100 (2008) 141601] gives \( f_+(0) = 0.964(5) \). They use 2+1 flavour of dynamical wall quarks.
  (recent update, G. Colangelo et al., European Physical Journal, C (2011) 71:1695 [FLAG report] gives 0.956 ± 0.008)

\[ f_0(M_K^2 - M_\pi^2) = \frac{F_K}{F_\pi} + \Delta_{CT} \]

\( \Delta_{CT} \simeq 0 \) to two-loops in chiral perturbation theory (J. Bijnens and P. Talavera, Nuclear Physics B 669 (2003) 341.)

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Difficult to estimate higher order corrections (to our knowledge not yet done in the literature).
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\[ \frac{F_K}{F_\pi} = 1.193 \pm 0.006 \] according to recent lattice evaluations (see e.g., L. Lellouch, arXiv:0902.4545; see also A. Bazavov et al. [MILC collaboration], arXiv:0910.2966, which uses 2+1 flavor with improved staggered quark action). Confirmed by S. Dürr et al. [BMW collaboration], arXiv:1001.4692. (FLAG report gives 1.193 ± 0.005 for 2+1 flavors averaged over three calculations, and 1.210 ± 0.018 with 2 flavors and a single calculation)
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An extremely interesting joint analysis of \( f_+(0) \) and \( \frac{F_K}{F_\pi} \) is by V. Bernard and E. Passemar, JHEP 1004 (2010) 001
ISTRA: Experimental setup at the IHEP 70 GeV proton synchrotron U-70. Secondary beam with about 25 GeV protons.
Experiments


- KLOE detector at DAFNE ($e^+e^-$ collider at 1.02 GeV) $K_L \rightarrow \pi\mu\nu$ analysis based on about 1.8 million events from 328 pb$^{-1}$. F. Ambrosino et al., JHEP 0712 (2007) 105.
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- **KTeV experiment at Fermilab**: 1.9 million $K_L$ electron and 1.5 million $K_L$ muono decays.
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Mushkelishvili-Omnès study of $\pi K$, $\pi K^*$, $K\rho$ and use of high statistics LASS experiment phase shifts used to produce the $\pi K$ vector form factor and compared with BELLE (B. Moussallam, European Physical Journal C 53 (2008) 401)
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Theoretical approaches

- Our work is motivated by the need to exploit in a complete and optimal way the available information.
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- Uses experimental information in such a way as to optimize all available inputs, and the modulus information only to evaluate an integral.
- Our phase and modulus data come from Moussallam, group of Jamin et al., and from BELLE.
Consider the QCD correlator

\[ \chi_0(Q^2) \equiv \frac{\partial}{\partial q^2} [q^2 \Pi_0] = \frac{1}{\pi} \int_{t_+}^{\infty} dt \frac{t \text{Im}\Pi_0(t)}{(t + Q^2)^2} , \]

\[ \text{Im}\Pi_0(t) \geq \frac{3}{2} \frac{t_+ t_-}{16\pi} \frac{[(t - t_+)(t - t_-)]^{1/2}}{t^3} |f_0(t)|^2 , \]

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Bounds can be obtained using analyticity to transform the problem, and to input values of the form factor and its derivatives at \( t = 0 \) and/or knowledge at various points in the analyticity region (method of unitarity bounds).
On the other hand, in pQCD when $Q \gg \Lambda_{\text{QCD}}, m_q, \alpha_S$, $\overline{MS}$ scheme.

$$\chi_0(Q^2) = \frac{3(m_s - m_u)^2}{8\pi^2 Q^2} \left[1 + 1.80\alpha_s + 4.65\alpha_s^2 + 15.0\alpha_s^3 + 57.4\alpha_s^4 \ldots\right].$$
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Reverse problem: to constrain $\lambda'_0, \lambda''_0$ and $f_0(\Delta_{K\pi})$ and $f_0(\Delta_{K\pi})$. 
Transforming via Conformal map

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The problem transformed

We can now use the conformal map to transform this to an integral that reads

\[ \frac{1}{2\pi} \int_0^{2\pi} |h(\exp(i\theta))|^2 \leq I_{PQCD} \]

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- For the case at hand:

\[
w(z) = \frac{3}{16\sqrt{2\pi}} \frac{M_K - M_\pi}{M_K + M_\pi} \sqrt{1 - z (1 + z)^{3/2}} \\
\times \left( \frac{(1 + z(-Q^2))^2}{(1 - z z(-Q^2))^2} \frac{(1 - z z(t_-))^{1/2}}{(1 + z(t_-))^{1/2}} \right),
\]

\[
h(z) = w(z)f_0(z).
\]
Analytic Interpolation Theory and Hardy Spaces

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Ideal setting for us since the original integral now is reduced to a series expansion on the Hardy Space and involves only the expansion coefficients.
Power series and origin of the bound

- Power series: \( h(z) = a_0 + a_1 z + a_2 z^2 + \ldots \) [Fourier series with non-negative powers of \( e^{i\theta} \)]. Guaranteed for such functions.
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- If the first $n$ coefficients of the form factor are known, a bound on the quantity of interest is obtained after a finite number of terms.
Some explicit expressions

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\[ a_1 = h'(0) = f_+(0)(w'(0) + \frac{2}{3}\langle r^2_{\pi K}\rangle t_+ w(0)), \]

\[ a_2 = \frac{h''(0)}{2!} = \frac{f_+(0)}{2} \left[ w(0) \left(-\frac{8}{3}\langle r^2_{\pi K}\rangle t_+ + 32 c t_+^2 \right) \right] \]
\[ + \frac{f_+(0)}{2} \left[ 2w'(0) \left(\frac{2}{3}\langle r^2_{\pi K}\rangle t_\pi \right) + w''(0) \right], \]
Improving the bounds

Improvement of the bound arises if $f_0(t)$ is known for some spacelike values of momenta corresponding to $z = x_i, i = 1, 2, 3, ...$
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- Improve the bound by using imposing constraints using Lagrange multipliers.
- Can also be improved by imposing phase of the form factor for timelike moment in a continuous region, $a \leq t \leq b$. 
Spacelike constraints

Spacelike constraints


- The case of two spacelike constraints is one where we solve:

\[
\begin{pmatrix}
I & a_0 & a_1 & a_2 & J_1 & J_2 \\
\begin{array}{rrrr}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & x_1 \\
0 & 0 & 1 & x_1^2 \\
1 & x_1 & x_1^2 & (1 - x_1^2)^{-1}
\end{array} & \begin{array}{rr}
1 & 1 \\
x_2 & x_2^2 \\
x_2 & x_2^2 \\
1 - x_2 & (1 - x_2)^{-1}
\end{array}
\end{pmatrix} = 0
\]

to obtain the bound, if \(a_i\) and \(J_i\) are known. Here \(I\) and \(J_i\) are known, and hence we can bound the \(a_i\)!
In the elastic region $t_+ \leq t \leq t_{\text{in}}$, the phase of the form factor is the scattering phase (Watson’s theorem). Can be included using Lagrange multipliers to obtain improved optimal constraints.
Inclusion of phase and modulus

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Idea is to defer the onset of the branch point to $t_{\text{in}}$


The present work is the only other known application of this powerful technique which is described in the following.
Consider the definition

\[ \mathcal{O}(t) = \exp \left( \frac{t}{\pi} \int_{t_+}^{\infty} dt \frac{\delta(t')}{{t'}(t' - t)} \right), \]

where \( \delta(t) \) is the \( I = 1/2 \) elastic S-wave \( K\pi \) scattering phase, in the elastic region and arbitrary Lipschitz continuous above \( t_{in} \) (viz., the phase and its first derivative are continuous).
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Since the Omnès function \( \mathcal{O}(t) \) fully accounts for the second Riemann sheet of the form factor, the function \( h(t) \), defined by

\[ f_0(t) = h(t) \mathcal{O}(t), \]

is real analytic in the \( t \)-plane with a cut only for \( t \geq t_{in} \).
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Extremely clever trick which makes the method very useful
The new conformal variable is now:

\[ z(t) = \frac{\sqrt{t_{\text{in}}} - \sqrt{t_{\text{in}} - t}}{\sqrt{t_{\text{in}}} + \sqrt{t_{\text{in}} - t}} \]

which maps the \( t \)-plane cut for \( t > t_{\text{in}} \) onto the unit disk \( |z| < 1 \), and

\[ h(z) = f_0(t(z)) w(z) \omega(z) [O(t(z))]^{-1}, \]
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Note that the Omnès function makes an appearance through its outer function \( (\omega(z)) \) and once as an inverse.
The new outer function is

\[ w(z) = \frac{3(M_k^2 - M_{\pi}^2)}{16\sqrt{2\pi}t_{\text{in}}} \sqrt{1 - z} \left(1 + z\right)^{3/2} \left(1 + z(-Q^2)\right)^2 \]

\[ \times \left(\frac{1 - z z(t_+)}{1 + z(t_+)}\right)^{1/2} \left(\frac{1 - z z(t_-)}{1 + z(t_-)}\right)^{1/2}, \]
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\times \frac{(1-zz(t_+))^{1/2}(1-zz(t_-))^{1/2}}{(1+z(t_+))^{1/2}(1+z(t_-))^{1/2}},
\]

An additional outer function now enters which is given by

\[
 \omega(z) = \exp \left( \frac{\sqrt{t_{\text{in}} - t}}{\pi} \int_{t_{\text{in}}}^{\infty} dt' \frac{\ln |O(t')|}{\sqrt{t' - t_{\text{in}}} (t' - t)} \right).
\]
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The input for the bound is now given by

\[ I = \chi_0(Q^2) - \frac{3}{2} \frac{t_+ t_-}{16\pi^2} \int_{t_+}^{t_{\text{in}}} dt \frac{[(t - t_+)(t - t_-)]^{1/2} |f_0(t)|^2}{t^2(t + Q^2)^2}. \]

Information of the modulus used in the integral.
Best results

- Our best constraints on the shape parameters of the scalar form factor
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- Comparison for results for vector form factor with no phase information, phase information, phase and modulus information
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- Region where zeros of the form factor are excluded
Best results for scalar shape parameters

![Graph showing best results for scalar shape parameters including data from Abouzaid, KTeV (2009) and Ambrosino, KLOE (2007).]
Best results for scalar shape parameters with CT

![Graph showing best results for scalar shape parameters with CT]
Scalar experiments – summary

-0.02 0 0.02 0.04 0.06

\( \lambda \)

- Ambrosino, KLOE (2007)
- Sciascia, Flavianet Kaon WG (2008)
- Amsler, PDG (2009)
- Abouzaid, KTeV (2009)

Unitarity and Analyticity Constraints... – p.30/42
Updated summary

Unitarity and Analyticity Constraints... – p.31/42
Results for vector shape parameters

Unitarity and Analyticity Constraints... – p.32/42
Best results for vector shape parameters

Zeros of form factors

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- Our method allows us search for zeros by using it as a SL constraint for both real and complex zeros
Influence of timelike zeros

\[ f_0(t) \]

Unitarity and Analyticity Constraints... – p.35/42
Influence of spacelike zeros

![Graph showing the influence of spacelike zeros on the function $f_0(t)$ against $t$ in GeV$^2$. The graph compares different scenarios with and without zero crossings, labeled as $T_0 = -0.1$ GeV$^2$ and $T_0 = -1$ GeV$^2$.](image)
Absence of zeros for the vector
Absence of zeros for the scalar including CT
Results

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- Tests the consistency of the determinations.
Recent work on inputs

- A. Bazavov et al., arXiv:1312.1228 [hep-ph] give $f_+(0) = 0.9704(32)$; S. Aoki et al., arXiv:1310.855 [hep-lat] $f_+(0) = 0.9624(36)$ (2+1 flavors) and $0.9595(9)$ (2 flavors)
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- V. Bernard, arXiv:1311.2569 [hep-ph] carries out a combined analysis of $\tau$ decays and $\pi K$ scattering to obtain slope and curvature of the vector form factor, easily accommodated in our bounds. Also obtains a new value for scalar form factor at the CT point consistent with prior determinations, but with larger errors, as this uses only dispersive methods.
$D - \pi$ form factors

- Uses in an essential way the heavy-light correlators computed by Chetyrkin and Steinhauser
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- Far less stringent than in the $\pi - K$ case
Conclusions

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