Theory status of inclusive $B$-meson decays

Mikołaj Misiak
University of Warsaw, Poland

1. Photon energy spectrum in $\bar{B} \rightarrow X_s\gamma$
2. Semileptonic decays $\bar{B} \rightarrow X_{c(u)}\ell\bar{\nu}$
3. NNLO QCD corrections to $\bar{B} \rightarrow X_s\gamma$
4. Rare semileptonic decays $\bar{B} \rightarrow X_s\ell^+\ell^-$
5. $\bar{B} \rightarrow X_s\nu\bar{\nu}$
6. Appendix: Rare leptonic decays $B_{s(d)} \rightarrow \ell^+\ell^-$

Rare decays are sensitive to new physics.

Semileptonic decays matter for the SM parameter determination.
Information on electroweak-scale physics in the $b \to s \gamma$ transition is encoded in an effective low-energy local interaction:

$$b \in \bar{B} \equiv (\bar{B}^0 \text{ or } B^-)$$

The inclusive $\bar{B} \to X_s \gamma$ decay rate is well approximated by the corresponding perturbative decay rate of the $b$-quark:

$$\Gamma(\bar{B} \to X_s \gamma)_{E_\gamma > E_0} = \Gamma(b \to X_s^p \gamma)_{E_\gamma > E_0} + \left(\text{non-perturbative effects} \right)_{(2 \pm 5)\%}$$

(Benzke et al., arXiv:1003.5012)

provided $E_0$ is large ($E_0 \sim m_b/2$) but not too close to the endpoint ($m_b - 2E_0 \gg \Lambda_{\text{QCD}}$).

Conventionally, $E_0 = 1.6 \text{ GeV} \simeq m_b/3$ is chosen.
SM estimate [hep-ph/0609232]:

\[ \mathcal{B}(\bar{B} \to X_s \gamma)_{E_\gamma > 1.6 \text{ GeV}}^{\text{SM}} = (3.15 \pm 0.23) \times 10^{-4} \]

Contributions to the total TH uncertainty (summed in quadrature):

5% non-perturbative, 3% from the interpolation in \( m_c \)

3% higher order \( \mathcal{O}(\alpha_s^3) \), 3% parametric

Experimental world average (HFAG, 2.08.2012):

\[ \mathcal{B}(\bar{B} \to X_s \gamma)_{E_\gamma > 1.6 \text{ GeV}}^{\text{EXP}} = (3.43 \pm 0.21 \pm 0.07) \times 10^{-4} \]

Experiment agrees with the SM at better than \( \sim 1\sigma \) level.

Uncertainties: TH \( \sim 7\% \), EXP \( \sim 6.5\% \).
The “raw” photon energy spectra in the inclusive measurements

The peaks are centered around \( \frac{1}{2}m_b \approx 2.35 \text{ GeV} \)
which corresponds to a two-body \( b \to s\gamma \) decay.

Broadening is due to (mainly):
• perturbative gluon bremsstrahlung,
• motion of the \( b \) quark inside the \( \bar{B} \) meson,
• motion of the \( \bar{B} \) meson in the \( \Upsilon(4S) \) frame.
The HFAG average includes the following measurements:

<table>
<thead>
<tr>
<th>Reference</th>
<th>Method</th>
<th># of $B\bar{B}$</th>
<th>$E_0$ [GeV]</th>
<th>$\mathcal{B} \times 10^4$ at $E_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CLEO [PRL 87 (2001) 251807]</td>
<td>inclusive</td>
<td>$9.70 \times 10^6$</td>
<td>2.0</td>
<td>$3.06 \pm 0.41 \pm 0.26$</td>
</tr>
<tr>
<td>BABAR [PRL 109 (2012) 191801]</td>
<td>inclusive</td>
<td>$3.83 \times 10^8$</td>
<td>1.8</td>
<td>$3.21 \pm 0.15 \pm 0.29 \pm 0.08$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.9</td>
<td>$3.00 \pm 0.14 \pm 0.19 \pm 0.06$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.0</td>
<td>$2.80 \pm 0.12 \pm 0.14 \pm 0.04$</td>
</tr>
<tr>
<td>BELLE [PRL 103 (2009) 241801]</td>
<td>inclusive</td>
<td>$6.57 \times 10^8$</td>
<td>1.7</td>
<td>$3.45 \pm 0.15 \pm 0.40$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.8</td>
<td>$3.36 \pm 0.13 \pm 0.25$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1.9</td>
<td>$3.21 \pm 0.11 \pm 0.16$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.0</td>
<td>$3.02 \pm 0.10 \pm 0.11$</td>
</tr>
<tr>
<td>BABAR [PRD 77 (2008) 051103]</td>
<td>inclusive with a hadronic tag</td>
<td>$2.32 \times 10^8$</td>
<td>1.9</td>
<td>$3.66 \pm 0.85 \pm 0.60$</td>
</tr>
<tr>
<td></td>
<td>(hadronic</td>
<td></td>
<td>2.0</td>
<td>$3.39 \pm 0.64 \pm 0.47$</td>
</tr>
<tr>
<td></td>
<td>decay of the</td>
<td></td>
<td>2.1</td>
<td>$2.78 \pm 0.48 \pm 0.35$</td>
</tr>
<tr>
<td></td>
<td>recoiling $B$ ($\bar{B}$)</td>
<td></td>
<td>2.2</td>
<td>$2.48 \pm 0.38 \pm 0.27$</td>
</tr>
<tr>
<td></td>
<td>events</td>
<td></td>
<td>2.3</td>
<td>$2.07 \pm 0.30 \pm 0.20$</td>
</tr>
<tr>
<td>BABAR [PRD 86 (2012) 052012]</td>
<td>semi-inclusive</td>
<td>$4.71 \times 10^8$</td>
<td>1.9</td>
<td>$3.29 \pm 0.19 \pm 0.48$</td>
</tr>
<tr>
<td>BELLE [PLB 511 (2001) 151]</td>
<td>semi-inclusive</td>
<td>$6.07 \times 10^6$</td>
<td>2.24→1.6</td>
<td>$3.69 \pm 0.58 \pm 0.46 \pm 0.60$</td>
</tr>
</tbody>
</table>
Comparison of the inclusive measurements of $\mathcal{B}(\bar{B} \to X_s\gamma)$ by CLEO, BELLE and BABAR for each $E_0$ separately

\[ \mathcal{B} \times 10^4 \text{ for each } E_0 \text{ [GeV]} \]

<table>
<thead>
<tr>
<th>Scheme</th>
<th>$E_\gamma &lt; 1.7$</th>
<th>$E_\gamma &lt; 1.8$</th>
<th>$E_\gamma &lt; 1.9$</th>
<th>$E_\gamma &lt; 2.0$</th>
<th>$E_\gamma &lt; 2.242$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kinetic</td>
<td>0.986 ± 0.001</td>
<td>0.968 ± 0.002</td>
<td>0.939 ± 0.005</td>
<td>0.903 ± 0.009</td>
<td>0.656 ± 0.031</td>
</tr>
<tr>
<td>Neubert SF</td>
<td>0.982 ± 0.002</td>
<td>0.962 ± 0.004</td>
<td>0.930 ± 0.008</td>
<td>0.888 ± 0.014</td>
<td>0.665 ± 0.035</td>
</tr>
<tr>
<td>Kagan-Neubert</td>
<td>0.988 ± 0.002</td>
<td>0.970 ± 0.005</td>
<td>0.940 ± 0.009</td>
<td>0.892 ± 0.014</td>
<td>0.643 ± 0.033</td>
</tr>
<tr>
<td>Average</td>
<td>0.985 ± 0.004</td>
<td>0.967 ± 0.006</td>
<td>0.936 ± 0.010</td>
<td>0.894 ± 0.016</td>
<td>0.655 ± 0.037</td>
</tr>
</tbody>
</table>

- Why do we need to extrapolate to lower $E_0$?
- Are the HFAG factors trustworthy?
Decoupling of $W, Z, t, H^0 \Rightarrow$ effective weak interaction Lagrangian:

$$L_{\text{weak}} \sim \sum C_i(\mu_b) Q_i$$

8 operators matter in the SM when the higher-order EW and/or CKM-suppressed effects are neglected:

\[
\begin{array}{c|c|c|c}
\text{current-current} & \text{photonic dipole} & \text{gluonic dipole} & \text{penguin} \\
\hline
\begin{array}{c}
c_L \quad b_L \\
Q_{1,2}
\end{array}
&
\begin{array}{c}
\gamma \\
Q_7
\end{array}
&
\begin{array}{c}
\gamma \\
Q_8
\end{array}
&
\begin{array}{c}
q \\
Q_{3,4,5,6}
\end{array}
\end{array}
\]

$$\Gamma(\bar{B} \rightarrow X_s \gamma)_{E_{\gamma}>E_0} = |C_7|^2 \Gamma_{77}(E_0) + \text{(other)}$$

Optical theorem:

$$d\Gamma_{77} \sim \text{Im} \{ \bar{B} \rightarrow X_s \gamma \} \equiv \text{Im} A$$

Integrating the amplitude $A$ over $E_\gamma$:

$$\int_{E_0}^{E_{\gamma}^{\text{max}}} dE_\gamma \sim \frac{\text{Im} E_\gamma}{E_{\gamma}^{\text{max}} - E_0} = \text{Im} \{ \bar{B} \rightarrow X_s \gamma \} \equiv \text{Im} A$$

Non-perturbative corrections to $\Gamma_{77}(E_0)$ form a series in $\frac{\Lambda_{\text{QCD}}}{m_b}$ and $\alpha_s$ that begins with

$$\frac{\mu_\pi^2}{m_b^2}, \frac{\mu_G^2}{m_b^2}, \frac{\rho_D^3}{m_b^3}, \frac{\rho_{LS}^3}{m_b^3}, \cdots; \frac{\alpha_s \mu_\pi^2}{(m_b-2E_0)^2}, \frac{\alpha_s \mu_G^2}{m_b(m_b-2E_0)^2}; \cdots,$$

where $\mu_\pi, \mu_G, \rho_D, \rho_{LS} = \mathcal{O}(\Lambda_{\text{QCD}})$ are extracted from the semileptonic $\bar{B} \rightarrow X_c e \bar{\nu}$ spectra and the $B-B^*$ mass difference.
The “hard” contribution to $\bar{B} \to X_s \gamma$

Goal: calculate the inclusive sum $\sum_{X_s} \left| C_7(\mu_b) \left< X_s \gamma | O_7 | \bar{B} \right> + C_2(\mu_b) \left< X_s \gamma | O_2 | \bar{B} \right> + \ldots \right|^2$

The “77” term in this sum is “hard”. It is related via the optical theorem to the imaginary part of the elastic forward scattering amplitude $\bar{B}(\vec{p} = 0)\gamma(\bar{q}) \to \bar{B}(\vec{p} = 0)\gamma(\bar{q})$:

$$\text{Im}\left\{ \bar{B} \bullet \bullet \bar{B} \right\} \equiv \text{Im} A$$

When the photons are soft enough, $m_{X_s}^2 = |m_B(m_B - 2E_\gamma)| \gg \Lambda^2 \Rightarrow \text{Short-distance dominance} \Rightarrow \text{OPE}$. However, the $\bar{B} \to X_s \gamma$ photon spectrum is dominated by hard photons $E_\gamma \sim \frac{m_b}{2}$.

Once $A(E_\gamma)$ is considered as a function of arbitrary complex $E_\gamma$, Im$A$ turns out to be proportional to the discontinuity of $A$ at the physical cut. Consequently,

$$\int_{1 \text{ GeV}}^{E_{\gamma}^{\text{max}}} dE_\gamma \text{ Im} A(E_\gamma) \sim \int_{\text{circle}} dE_\gamma A(E_\gamma).$$

Since the condition $|m_B(m_B - 2E_\gamma)| \gg \Lambda^2$ is fulfilled along the circle, the OPE coefficients can be calculated perturbatively, which gives

$$A(E_\gamma)|_{\text{circle}} \sim \sum_j \left[ \frac{F_{\text{polynomial}}(2E_\gamma/m_b)}{m_b^{n_j}(1 - 2E_\gamma/m_b)^{k_j}} + \mathcal{O}(\alpha_s(\mu_{\text{hard}})) \right] \left< \bar{B}(\vec{p} = 0) | Q_j^{(j)} | \bar{B}(\vec{p} = 0) \right>.$$ 

Thus, contributions from higher-dimensional operators are suppressed by powers of $\Lambda/m_b$.

At $(\Lambda/m_b)^0$: $\langle \bar{B}(\vec{p}) | \bar{b} \gamma^\mu b | \bar{B}(\vec{p}) \rangle = 2p^\mu \Rightarrow \Gamma(\bar{B} \to X_s \gamma) = \Gamma(b \to X_s \text{ parton} \gamma) + \mathcal{O}(\Lambda/m_b)$.

At $(\Lambda/m_b)^1$: Nothing! All the possible operators vanish by the equations of motion.

At $(\Lambda/m_b)^2$:

$$\langle \bar{B}(\vec{p}) | \bar{b}_v D^\mu D_\mu b_v | \bar{B}(\vec{p}) \rangle \sim m_B \mu^2,$$

$$\langle \bar{B}(\vec{p}) | \bar{b}_v g_s G_{\mu\nu} \sigma^{\mu\nu} b_v | \bar{B}(\vec{p}) \rangle \sim m_B \mu^2_G,$$

The HQET heavy-quark field $b_v(x)$ is defined by $b_v(x) = \frac{1}{2}(1 + \psi) b(x) \exp(im_b v \cdot x)$ with $v = p/m_B$. 

---

The $\mathcal{O}\left(\frac{\alpha_s \mu^2_{\pi}}{(m_b - 2E_0)^2}\right)$ and $\mathcal{O}\left(\frac{\alpha_s \mu^2_G}{m_b(m_b - 2E_0)}\right)$ corrections


$$\Gamma_{77}(E_0) = \Gamma_{77}^{\text{tree}} \left\{ 1 + \text{(pert. corrections)} - \frac{\mu^2_{\pi}}{2m_b^2} \left[ 1 + \frac{\alpha_s}{\pi} \left( f_1(E_0) - \frac{4}{3} \ln \frac{\mu}{m_b} \right) \right] - \frac{3\mu^2_G(\mu)}{2m_b^2} \left[ 1 + \frac{\alpha_s}{\pi} \left( f_2(E_0) + \frac{1}{6} \ln \frac{\mu}{m_b} \right) \right] \right\}$$
When \((m_b - 2E_0) \sim \Lambda \equiv \Lambda_{QCD}\), no OPE can be applied.

**Local operators → Non-local operators**

**Non-perturbative parameters → Non-perturbative functions**

\[
\frac{d}{dE_\gamma} \Gamma_{77} = N H(E_\gamma) \int_0^{MB-2E_\gamma} dk \ P(MB-2E_\gamma-k) \ F(k) + \mathcal{O} \left( \frac{\Lambda}{m_b} \right)
\]

Photon spectra from models of \(F(k)\) [Ligeti, Stewart, Tackmann, arXiv:0807.1926]

The function \(F(k)\) is:
- perturbatively related to the standard shape function \(S(\omega)\),
- exponentially suppressed for \(k \gg \Lambda\),
- positive definite,
- constrained by measured moments of the \(\bar{B} \to X_c e \bar{\nu}\) spectrum (local OPE),
- constrained by measured properties of the \(\bar{B} \to X_u e \bar{\nu}\) and \(\bar{B} \to X_s \gamma\) spectra (not imposed in the plot).
Upgrading the HFAG factors by fitting $F(k)$ to data:

- The SIMBA Collaboration [arXiv:1101.3310] (work in progress)
  
  $$F(k) = \frac{1}{\lambda} \left[ \sum_{n=0}^{\infty} c_n f_n \left( \frac{k}{\lambda} \right) \right]^2, \quad f_n - \text{basis functions. Truncate and fit.}$$

- Another way: $F(k) = A(k)B(k)$ and use the SIMBA approach for $B(k)$.

Why do we need to upgrade the HFAG factors?

- The old models (Kagan-Neubert 1998, ...) are not generic enough (too few parameters).
- Inclusion of $\mathcal{O} \left( \frac{\Lambda}{m_b} \right)$ effects and and taking other operators ($Q_i \neq Q_7$) into account is necessary [Benzke, Lee, Neubert, Paz, arXiv:1003.5012].

What about just fitting $C_7$ without extrapolation to any particular $E_0$?

- Fine, but measurements at low $E_0$ (even less precise) are still going to be crucial for constraining the parameter space.
- The fits are going to give the extrapolation factors anyway. Publishing them is necessary for cross-checks/upgrades by other groups.
Non-perturbative effects in the presence of other operators \((Q_i \neq Q_7)\)

\[
\frac{d}{dE_\gamma} \Gamma(\bar{B} \to X_s \gamma) = (\Gamma_{77}\text{-like term}) + \tilde{N}E_\gamma^3 \sum_{i \leq j} \text{Re} \left( C_i^* C_j \right) F_{ij}(E_\gamma).
\]

**Remarks:**

- The SCET approach is valid for large \(E_\gamma\) only. It is fine for \(E_\gamma > E_0 \sim \frac{1}{3} m_b \simeq 1.6 \text{ GeV}\). Lower cutoffs are academic anyway.

- For such \(E_0\), non-perturbative effects in the integrated decay rate are estimated to remain within 5%. They scale like:

  - \(\frac{\Lambda^2}{m_b^2}, \frac{\Lambda^2}{m_c^2}\) (known),
  
  - \(\frac{\Lambda}{m_b} \frac{V_{us}^* V_{ub}}{V_{ts}^* V_{tb}}\) (negligible),

  - \(\frac{\Lambda}{m_b}, \frac{\Lambda^2}{m_b^2}, \alpha_s \frac{\Lambda}{m_b}\) but suppressed by tails of subleading shape functions ("27"),

  - \(\alpha_s \frac{\Lambda}{m_b}\) to be constrained by future measurements of the isospin asymmetry ("78"),

  - \(\alpha_s \frac{\Lambda}{m_b}\) but suppressed by \(Q_d^2 = \frac{1}{9}\) ("88").

- **Extrapolation factors?** Tails of subleading functions are less important for them.
Semileptonic $\bar{B}$-meson decays

Inclusive $\bar{B} \rightarrow X\ell\bar{\nu}$ decay rates are well approximated by the corresponding $b$-quark decay rates, similarly to $\Gamma_{77}$ in the radiative decay. For instance,

$$\Gamma[\bar{B} \rightarrow X_u e\bar{\nu}] = \Gamma[b \rightarrow X^\text{parton}_u e\bar{\nu}] + \mathcal{O}\left(\frac{\Lambda^2}{m_b^2}\right),$$

where

$$\mathcal{O}\left(\frac{\Lambda^2}{m_b^2}\right) = \sum a_k \frac{1}{m_k} \langle \bar{B}|Q_k|\bar{B}\rangle.$$ 

Similarly:

$$\Gamma[\bar{B} \rightarrow X_c e\bar{\nu}] = \Gamma[b \rightarrow X^\text{parton}_c e\bar{\nu}] + \mathcal{O}\left(\frac{\Lambda^2}{(m_b-m_c)^2}\right),$$

See, e.g.,
The semileptonic decay rate, as well as moments of the lepton energy spectra can be calculated from the following correlator:

\[ W_{\mu\nu}(v, q) = \text{Im} \frac{2i}{\pi M_B} \int d^4 x \ e^{-i q x} \langle \bar{B} | T J_\mu^\dagger(x) J_\nu(0) | \bar{B} \rangle, \]

where \( J_\mu = \bar{c} \gamma_\mu P_L b \) (in the \( b \rightarrow c \ell \bar{\nu} \) case), \( q = k_\ell + k_\nu \) and \( p_B = M_B v \).

Decomposition:

\[ m_b W^{\mu\nu} = -W_1 g^{\mu\nu} + W_2 v^\mu v^\nu + i W_3 \varepsilon^{\mu\nu\rho\sigma} v_\rho \hat{q}_\sigma + W_4 \hat{q}^\mu \hat{q}^\nu + W_5 (v^\mu \hat{q}^\nu + v^\nu \hat{q}^\mu), \]

where \( \hat{q} = q/m_b \). The coefficients \( W_i \) are functions of \( \hat{q}^2 \), \( v \cdot \hat{q} \) and \( m_c/m_b \).

OPE \( \Rightarrow \) Perturbative expansion in powers of \( \alpha_s \) and \( \Lambda_{QCD}/m_b \):

\[ W_i = W_i^{(0)} + \frac{\mu_\pi^2}{2m_b^2} W_i^{(\pi,0)} + \frac{\mu_G^2}{2m_b^2} W_i^{(G,0)} + \mathcal{O} \left( \frac{\Lambda^3}{m_b^3} \right) + \mathcal{O} \left( \frac{\Lambda^4}{m_b^4} \right) + \mathcal{O} \left( \frac{\Lambda^5}{m_b^5} \right) + \]

\[ + \frac{\alpha_s}{\pi} \left[ C_F W_i^{(1)} + C_F \frac{\mu_\pi^2}{2m_b^2} W_i^{(\pi,1)} + \frac{\mu_G^2}{2m_b^2} W_i^{(G,1)} \right] + \cdots \]

Here:

\[ \mu_\pi^2 = \frac{1}{2M_B} \langle \bar{B} | b_v (i \not{D})^2 b_v | \bar{B} \rangle, \]

\[ \mu_\pi^2 = -\frac{g_s}{4M_B} \langle \bar{B} | b_v G_{\mu\nu} \sigma^{\mu\nu} b_v | \bar{B} \rangle. \]

Caution: \( W_i^{(...)} \neq W_i^{(...)}_{\text{pert}} \) before integration over \( v \cdot \hat{q} \).
Numerical expressions for the semileptonic $\bar{B} \to X_c \ell \bar{\nu}$ decay width $\Gamma$, the mean lepton energy $\langle E_\ell \rangle$ and the variance (second central moment) $\ell_2 \equiv \langle (E_\ell - \langle E_\ell \rangle)^2 \rangle$

from [A. Alberti, P. Gambino and S. Nandi, arXiv:1311.7381]:

$$\Gamma = \Gamma_0 \left[ 1 - 1.11 \frac{\alpha_s}{\pi} - \left( \frac{1}{2} - 0.99 \frac{\alpha_s}{\pi} \right) \frac{\mu_\pi^2}{m_b^2} - \left( 1.94 + 3.46 \frac{\alpha_s}{\pi} \right) \frac{\mu_G^2(m_b)}{m_b^2} \right] + \cdots,$$

$$\langle E_\ell \rangle = 1.41 \text{GeV} \left[ 1 - 0.01 \frac{\alpha_s}{\pi} + \left( \frac{1}{2} - 0.44 \frac{\alpha_s}{\pi} \right) \frac{\mu_\pi^2}{m_b^2} - \left( 1.19 + 3.21 \frac{\alpha_s}{\pi} \right) \frac{\mu_G^2(m_b)}{m_b^2} \right] + \cdots,$$

$$\ell_2 = 0.183 \text{ GeV}^2 \left[ 1 - 0.24 \frac{\alpha_s}{\pi} + \left( 4.98 - 3.89 \frac{\alpha_s}{\pi} \right) \frac{\mu_\pi^2}{m_b^2} - \left( 2.89 + 7.01 \frac{\alpha_s}{\pi} \right) \frac{\mu_G^2(m_b)}{m_b^2} \right] + \cdots.$$

Comparing such quantities to the measured ones (BaBar, Belle, CDF, CLEO, DELPHI) one determines $|V_{cb}|$, quark masses and the non-perturbative parameters $\mu_\pi^2$, $\mu_G^2$, $\cdots$ in a single fit. Information on the quark masses from $e^+e^-$ data, as well on $\mu_G^2$ from $M_{B^*} - M_B$ can be taken into account, too.


$\mathcal{O}(1)$, $\mathcal{O}(\alpha_s)$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}\left(\frac{\Lambda^2}{m_b^2}\right)$, $\mathcal{O}\left(\frac{\Lambda^3}{m_b^3}\right)$ but no $\mathcal{O}\left(\alpha_s \frac{\Lambda^2}{m_b^2}\right)$, or $\mathcal{O}\left(\frac{\Lambda^{4,5}}{m_b^{4,5}}\right)$ yet.

In the latter case, lattice QCD might help in the future, even with very rough estimates.
Sample results of the recent fit to the semileptonic data

Current status of $|V_{ub}|$ and $|V_{cb}|$ determinations, as summarized by the Flavour Lattice Averaging Group (FLAG), arXiv:1310.8555.

Tensions:  
$\sim 2.3\sigma$ for $|V_{cb}|$

$\sim 3\sigma$ for $|V_{ub}|$ (without $B \rightarrow \tau\nu$)

Figure 23: Comparison of the results for $|V_{ub}|$ and $|V_{cb}|$ obtained from lattice methods with non-lattice determinations based on inclusive semileptonic $B$ decays. In the left plot, the results denoted by squares are from leptonic decays, while those denoted by triangles are from semileptonic decays. The grey band indicates our $N_f = 2 + 1$ average.
NNLO QCD corrections to $\bar{B} \to X_s \gamma$

The relevant perturbative quantity:

$$\frac{\Gamma[b\to X_s \gamma|E_\gamma>E_0]}{|V_{cb}/V_{ub}|^2 \Gamma[b\to X_u e\bar{\nu}]} = \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{em}}{\pi} \sum_{i,j} C_i C_j K_{ij} P(E_0)$$

Expansions of the Wilson coefficients and $K_{ij}$:

$$C_i(\mu_b) = C_i^{(0)}(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_i^{(1)}(\mu_b) + \left( \frac{\alpha_s(\mu_b)}{4\pi} \right)^2 C_i^{(2)}(\mu_b) + \ldots$$

$$K_{ij} = K_{ij}^{(0)} + \frac{\alpha_s(\mu_b)}{4\pi} K_{ij}^{(1)} + \left( \frac{\alpha_s(\mu_b)}{4\pi} \right)^2 K_{ij}^{(2)} + \ldots \quad \mu_b \sim \frac{m_b}{2}$$

Most important at the NNLO: $K_{77}^{(2)}$, $K_{27}^{(2)}$ and $K_{17}^{(2)}$.

They depend on $\frac{\mu_b}{m_b}$, $\frac{E_0}{m_b}$ and $r = \frac{m_c}{m_b}$. 
Evaluation of $K_{27}^{(2)}$ and $K_{17}^{(2)}$ for $m_c = 0$:

[M. Czakon, P. Fiedler, T. Huber, MM, T. Schutzmeier, M. Steinhauser, to be published]
Master integrals and differential equations:

<table>
<thead>
<tr>
<th></th>
<th>$n_D$</th>
<th>$n_{OS}$</th>
<th>$n_{eff}$</th>
<th>$n_{massless}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-particle cuts</td>
<td>292</td>
<td>92</td>
<td>143</td>
<td>9</td>
</tr>
<tr>
<td>3-particle cuts</td>
<td>267</td>
<td>54</td>
<td>110</td>
<td>11</td>
</tr>
<tr>
<td>4-particle cuts</td>
<td>292</td>
<td>17</td>
<td>37</td>
<td>7</td>
</tr>
</tbody>
</table>

\[
\frac{d}{dz} I_i(z) = \sum_j R_{ij}(z) I_j(z), \quad z = \frac{p^2}{m_b^2}.
\]

Boundary conditions in the vicinity of $z = 0$:
Massless integrals for the boundary conditions:

<table>
<thead>
<tr>
<th>2PCuts</th>
<th>3PCuts</th>
<th>4PCuts</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="2L2C1" /></td>
<td><img src="image1" alt="3L2C1" /></td>
<td><img src="image1" alt="4L4C1" /></td>
</tr>
<tr>
<td><img src="image1" alt="4L2C1" /></td>
<td><img src="image1" alt="4L3C1" /></td>
<td><img src="image1" alt="4L4C2" /></td>
</tr>
<tr>
<td><img src="image1" alt="4L2C2" /></td>
<td><img src="image1" alt="4L3C2" /></td>
<td><img src="image1" alt="4L4C3" /></td>
</tr>
<tr>
<td><img src="image1" alt="4L2C3" /></td>
<td><img src="image1" alt="4L3C3" /></td>
<td><img src="image1" alt="4L4C4" /></td>
</tr>
<tr>
<td><img src="image1" alt="4L2C4" /></td>
<td><img src="image1" alt="4L3C4" /></td>
<td><img src="image1" alt="4L4C5" /></td>
</tr>
<tr>
<td><img src="image1" alt="4L2C5" /></td>
<td><img src="image1" alt="4L3C5" /></td>
<td><img src="image1" alt="4L4C6" /></td>
</tr>
<tr>
<td><img src="image1" alt="4L2C6" /></td>
<td><img src="image1" alt="4L3C6" /></td>
<td><img src="image1" alt="4L4C7" /></td>
</tr>
<tr>
<td><img src="image1" alt="4L2C7" /></td>
<td><img src="image1" alt="4L3C7" /></td>
<td><img src="image1" alt="4L4C8" /></td>
</tr>
</tbody>
</table>
Results for the NNLO corrections:

\[ K_{27}^{(2)}(r, E_0) = A_2 + F_2(r, E_0) + 3f_q(r, E_0) + f_b(r) + f_c(r) + \frac{8}{3}\phi_{27}^{(1)}(r, E_0) \ln r \]
\[
+ \left[ (4L_c - x_m) r \frac{d}{dr} + x_mE_0 \frac{d}{dE_0} \right] f_{NLO}(r, E_0) + \frac{416}{81}x_m \\
+ \left( \frac{10}{3}K_{27}^{(1)} - \frac{2}{3} K_{47}^{(1)} - \frac{208}{81}K_{77}^{(1)} - \frac{35}{27}K_{78}^{(1)} - \frac{254}{81} \right) L_b - \frac{5948}{729}L_b^2,
\]

\[ K_{17}^{(2)}(r, E_0) = -\frac{1}{6}K_{27}^{(2)}(r, E_0) + A_1 + F_1(r, E_0) + \left( \frac{94}{81} - \frac{3}{2}K_{27}^{(1)} - \frac{3}{4}K_{78}^{(1)} \right) L_b - \frac{34}{27}L_b^2,
\]

where \( F_i(0, 0) \equiv 0, \ A_1 \approx 22.605, \ A_2 \approx -37.314 \) from the present calculation.

Correction due to \( \mathcal{O}(\epsilon) \) term in one of the master integrals. Currently being cross-checked.

Next, we interpolate in \( m_c \) by assuming that \( F_i(r, 0) \) are linear combinations of \( f_q(r, 0), \ f_{NLO}(r, 0), \ r \frac{d}{dr} f_{NLO}(r, 0) \) and a constant term.

The known large-\( r \) behaviour of \( F_i \) [hep-ph/0609241] and the condition \( F_i(0, 0) \equiv 0 \) fix these linear combinations in a unique manner.
Interferences not involving the photonic dipole operator are treated as follows:

\[ K_{22} : \]
(and analogous \( K_{11} \) & \( K_{12} \))

\[ K_{28} : \]
(and analogous \( K_{18} \))

\[ K_{88} : \]

Two-particle cuts are known (just \( |NLO|^2 \)).

Three- and four-particle cuts are known in the BLM approximation only. The NLO+(NNLO BLM) corrections are not big (+3.8%).
Incorporating other perturbative contributions evaluated after the previous phenomenological analysis in hep-ph/0609232:

1. Four-loop mixing (current-current) → (gluonic dipole)

2. Diagrams with massive quark loops on the gluon lines

3. Complete interference (photonic dipole)–(gluonic dipole)
   H. M. Asatrian, T. Ewerth, A. Ferroglia, C. Greub and G. Ossola,

4. New BLM corrections to contributions from 3-body and 4-body final states for interferences not involving the photonic dipole:

5. LO contributions from $b \rightarrow s\gamma q\bar{q}$, ($q = u, d, s$) from the four quark operators (“penguin” ones or CKM-suppressed ones).

Taking into account new non-perturbative analyses:


Updating the parameters (Parametric uncertainties go down to 2.4%)

Relative NLO and NNLO QCD corrections to $\mathcal{B}(\bar{B} \to X_s \gamma)$ and their dependence on $m_c/m_b$
The following vertices $Q_i$ matter for $b \to s\gamma$ and $b \to sl^+l^-$:

(SM – only the red ones)

$Q_{1,2} = \begin{array}{c}
\text{c}_L \\
\text{b}_L \\
\text{c}_L \\
\text{s}_L \\
\end{array}$

$Q_7 = \begin{array}{c}
\text{b}_R \\
\sigma \\
\text{s}_L \\
\end{array}$

$Q'_7 = \begin{array}{c}
\text{b}_L \\
\sigma \\
\text{s}_R \\
\end{array}$

$Q_{3,4,5,6} = \begin{array}{c}
\text{q} \\
\text{b}_L \\
\text{q} \\
\text{s}_L \\
\end{array}$

$Q_8 = \begin{array}{c}
\text{b}_R \\
\sigma \\
\text{s}_L \\
\end{array}$

$Q'_8 = \begin{array}{c}
\text{b}_L \\
\sigma \\
\text{s}_R \\
\end{array}$

$Q_9 = \begin{array}{c}
\text{b}_L \\
\gamma_\mu \\
\text{s}_L \\
\end{array}$

$Q'_9 = \begin{array}{c}
\text{b}_R \\
\gamma_\mu \\
\text{s}_R \\
\end{array}$

$Q_{10} = \begin{array}{c}
\text{b}_L \\
\gamma_{\mu\gamma_5} \\
\text{s}_L \\
\end{array}$

$Q'_{10} = \begin{array}{c}
\text{b}_R \\
\gamma_{\mu\gamma_5} \\
\text{s}_R \\
\end{array}$

$Q_S = \begin{array}{c}
\text{b}_R \\
\gamma_5 \\
\text{s}_L \\
\end{array}$

$Q'_S = \begin{array}{c}
\text{b}_L \\
\gamma_5 \\
\text{s}_R \\
\end{array}$

$Q_P = \begin{array}{c}
\text{b}_R \\
\gamma_5 \\
\text{s}_L \\
\end{array}$

$Q'_P = \begin{array}{c}
\text{b}_L \\
\gamma_5 \\
\text{s}_R \\
\end{array}$

$Q_T = \begin{array}{c}
\text{b}_R \\
\sigma_{\mu\nu} \\
\text{s}_L \\
\end{array}$

$Q'_T = \begin{array}{c}
\text{b}_L \\
\sigma_{\mu\nu} \\
\text{s}_R \\
\end{array}$

Assumption: no relevant NP effects in the 4-quark operators.
Dilepton mass spectrum in $\bar{B} \rightarrow X_s l^+ l^-$ ($l = e$ or $\mu$)

HFAG average (peak regions removed):
\[
\mathcal{B}(\bar{B} \rightarrow X_s l^+ l^-) = (3.66 \pm 0.77) \times 10^{-6}
\]

with non-perturbative $c\bar{c}$ using “naive” factorization
[F. Krüger, L.M. Sehgal hep-ex/9603237]

\[
\frac{m_b \frac{d\mathcal{B}(\bar{B} \rightarrow X_s l^+ l^-)}{dm_{l^+ l^-}} \times 10^5}{J/\psi \psi'}\text{perturbative}
\]

\[
d\Gamma(\bar{B} \rightarrow X_s l^+ l^-) = \frac{G_F^2 m_{b,\text{pole}}^5 |V_{ts}^* V_{tb}|^2}{48\pi^3} \left(\frac{\alpha_{\text{em}}}{4\pi}\right)^2 (1 - \hat{S})^2 \times
\]
\[
\left\{ (1 + 2\hat{S}) (|C_{9}^{\text{eff}}(\hat{S})|^2 + |C_{10}^{\text{eff}}(\hat{S})|^2) + \left(4 + \frac{8}{\hat{S}}\right) |C_{7}^{\text{eff}}(\hat{S})|^2 + 12 \text{Re} \left(C_{7}^{\text{eff}}(\hat{S}) C_{9}^{\text{eff}*}(\hat{S})\right) \right\} + R.
\]

$C_{i}^{\text{eff}}(\hat{S}) = C_{i}(\mu_b) + (\text{loop corrections})(\hat{S})$.

$R$ stands for small bremsstrahlung contributions and for the non-perturbative corrections.
Sample bounds on the Wilson coefficients from a recent global fit to exclusive and inclusive $b \to s$ observables


Constraints from $\mathcal{B}(\bar{B} \to X_s \gamma)$, $\mathcal{B}(\bar{B} \to X_s l^+l^-)$, $\mathcal{B}(B_s \to \mu^+\mu^-)$,

$\bar{B} \to K^*\gamma$: $\mathcal{B}$, $S$ (↔ mixing-induced CP asymmetry) and $C$,

$\bar{B} \to K^*l^+l^-$: $\mathcal{B}$ and angular observables $\{A_{FB}, F_L, A_T^{(2)}, P_{4,5,6}'\}$.

98 constraints, 28 nuisance parameters.

Next on the list:

$\mathcal{B}(\bar{B} \to X_s l^+l^-) = (6.73^{+0.70}_{-0.64}[\text{stat}]^{+0.34}_{-0.25}[\text{syst}] \pm 0.50[\text{model}]) \times 10^{-6},$

\[ \mathcal{B}(\bar{B} \rightarrow X_s \nu \bar{\nu}) \]

On the theory side – as clean as the semileptonic decay \[ \Rightarrow \] uncertainty is dominated by \(|V_{cb}|^2\). Ratios can be considered.

On the experimental side – even at Belle-II with 50 ab\(^{-1}\), only the exclusive \[ B \rightarrow K^{(*)} \nu \bar{\nu} \] modes are mentioned in the TDR with accuracies at the level of 30%.

**Belle-II, arXiv:1011.0352**

---

<table>
<thead>
<tr>
<th>Observable</th>
<th>Belle 2006 ((\sim 0.5) ab(^{-1}))</th>
<th>Belle II/SuperKEKB (5 ab(^{-1}))</th>
<th>LHCb(^{\dagger}) (2 fb(^{-1}))</th>
<th>LHCb(^{\dagger}) (10 fb(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hadronic (b \rightarrow s) transitions</td>
<td>[ \Delta S_{\phi K^0} ] 0.22</td>
<td>0.073</td>
<td>0.029</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>[ \Delta S_{\eta' K^0} ] 0.11</td>
<td>0.038</td>
<td>0.020</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ \Delta S_{K_S^0 K_S^0 K_S^0} ] 0.33</td>
<td>0.105</td>
<td>0.037</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ \Delta A_{\pi^0 K_S^0} ] 0.15</td>
<td>0.072</td>
<td>0.042</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ A_{\phi \phi K^+} ] 0.17</td>
<td>0.05</td>
<td>0.014</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ \phi_{eff}^{\text{Dalitz}} (\phi K_S) ] Dalitz 3.3°</td>
<td>1.5°</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Radiative/electroweak (b \rightarrow s) transitions</td>
<td>[ S_{K_S^0 \pi^0 \gamma} ] 0.32</td>
<td>0.10</td>
<td>0.03</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ \mathcal{B}(\bar{B} \rightarrow X_s \gamma) ] 13%</td>
<td>7%</td>
<td>6%</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ A_{CP}(B \rightarrow X_s \gamma) ] 0.058</td>
<td>0.01</td>
<td>0.005</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ C_9 \text{ from } A_{FB}(B \rightarrow K^* \ell^+ \ell^-) ] -</td>
<td>11%</td>
<td>4%</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ C_{10} \text{ from } A_{FB}(B \rightarrow K^* \ell^+ \ell^-) ] -</td>
<td>13%</td>
<td>4%</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ C_7/C_9 \text{ from } A_{FB}(B \rightarrow K^* \ell^+ \ell^-) ] -</td>
<td>5%</td>
<td>7%</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ R_K ] 0.07</td>
<td>0.02</td>
<td>0.043</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ \mathcal{B}(B^+ \rightarrow K^+ \nu \bar{\nu}) ] (\dagger \dagger &lt; 3 \mathcal{B}_{\text{SM}})</td>
<td>30%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>[ \mathcal{B}(B^0 \rightarrow K^{*0} \nu \bar{\nu}) ] (\dagger \dagger &lt; 40 \mathcal{B}_{\text{SM}})</td>
<td>35%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Radiative/electroweak (b \rightarrow d) transitions</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

---

\(\dagger\) denotes signal over background. \(\dagger \dagger\) denotes 90% CL upper limit.
Summary

- Determination of $|V_{cb}|$ from inclusive semileptonic $B$ decays is currently limited by theory uncertainties. Rough estimates of higher-dimensional operator matrix elements would help.

- The observed photon energy spectrum in $\bar{B} \rightarrow X_s \gamma$ can be used to constrain the shape function parameters, leading to a more precise determination of the total rate. At the same time, improvement in the inclusive determination of $|V_{ub}|$ can be achieved (combined fits).

- Dominant NNLO corrections to $\mathcal{B}(\bar{B} \rightarrow X_s \gamma)$ will soon be known not only in the large $m_c$ limit, but also at $m_c = 0$. If the current result survives, no reduction of uncertainties with respect to the 2006 estimate is expected, except for the parametric one.

- Measurements of $\bar{B} \rightarrow X_s \ell^+ \ell^-$ have been combined in several global fits with all the data on $b \rightarrow s$ transitions. The accuracy with which $C_9$ and $C_{10}$ are determined becomes now comparable to the one of $C_7$.

- Inclusive $\bar{B} \rightarrow X_s \nu \bar{\nu}$ branching ratio is very clean on the theoretical side. However, feasibility studies for Belle-II have been focused so far on the exclusive modes only, for which accuracies at the 30% level can be reached.
APPENDIX
$B_s \to \mu^+ \mu^-$ — the flavour physics highlight of the LHC

- It is a strongly suppressed, loop-generated process in the SM. Its average time-integrated branching ratio (with final-state photon bremsstrahlung included) reads:

$$\overline{\mathcal{B}}_{\text{SM}} = (3.65 \pm 0.23) \times 10^{-9}$$


- It is very sensitive to new physics even in models with Minimal Flavour Violation (MFV). Enhancements by orders of magnitude are possible even when constraints from all the other measurements are taken into account.

- It has a clear experimental signature: **PEAK** in the dimuon invariant mass.

- Recently measured branching ratios

$$\overline{\mathcal{B}}_{\text{exp}} = \begin{cases} (2.9^{+1.1}_{-1.0}) \times 10^{-9}, & \text{LHCb} \ [\text{Phys. Rev. Lett. 111 (2013) 101805}] \\ (3.0^{+1.0}_{-0.9}) \times 10^{-9}, & \text{CMS} \ [\text{Phys. Rev. Lett. 111 (2013) 101804}] \end{cases}$$

Combined: $$\overline{\mathcal{B}}_{\text{exp}} = (2.9 \pm 0.7) \times 10^{-9}$$

[ CMS-PAS-BPH-13-007, LHCb-CONF-2013-012 ]
Operators (dim 6) that matter for $B_s \to \mu^+\mu^-$ read

$$Q_A = (\bar{b} \gamma^\alpha \gamma_5 s) (\bar{\mu} \gamma^\alpha \gamma_5 \mu)$$

$$Q_S(P) = (\bar{b} \gamma_5 s) (\bar{\mu}(\gamma_5) \mu) = \frac{i(\bar{b} \gamma^\alpha \gamma_5 s) \partial_\alpha (\bar{\mu}(\gamma_5) \mu)}{m_b + m_s} + E + T$$

necessary non-perturbative input:

$$\langle 0 | \bar{b} \gamma^\alpha \gamma_5 s | B_s(p) \rangle = ip^\alpha f_{B_s}$$

recent lattice determinations of the $B_s$-meson decay constant:

$$f_{B_s} = \begin{cases} 
225.0(4.0) \text{ MeV}, & \text{HPQCD (r), arXiv:1110.4510} \\
224.0(5.0) \text{ MeV}, & \text{HPQCD (nr), arXiv:1302.2644} \\
234.0(6.0) \text{ MeV}, & \text{ROME, arXiv:1212.0301} \\
242.0(9.5) \text{ MeV}, & \text{FNAL/MILC, arXiv:1112.3051} \\
232(10) \text{ MeV}, & \text{ETM, arXiv:1107.1441} \\
219(12) \text{ MeV}, & \text{ALPHA, arXiv:1210.6524} 
\end{cases}$$

average time-integrated branching ratio:

$$\overline{\mathcal{B}}(B_s \to \mu^+\mu^-) = \frac{|N|^2 B_{B_s}^3 f_{B_s}^2}{8\pi \Gamma_s^H} \beta \left( |r C_A - u C_P|^2 F_P + |u \beta C_S|^2 F_S \right) + O(\alpha_{em}),$$

where

$$N = \frac{V_{tb}^* V_{ts} G_F^2 M_W^2}{\pi^2}, \quad r = \frac{2m_\mu}{M_{B_s}}, \quad \beta = \sqrt{1 - r^2}, \quad u = \frac{M_{B_s}}{m_b + m_s},$$

$$F_P = 1 - \frac{\Delta \Gamma_s}{\Gamma_s^L} \sin^2 \left[ \frac{1}{2} \phi_{NP} + \arg(r C_A - u C_P) \right] \quad \text{SM CP} \quad \frac{\Gamma^H_s}{\Gamma_s^L} \quad \text{derived following [K. de Bruyn et al., Phys. Rev. Lett. 109 (2012) 041801]}$$

$$F_S = 1 - \frac{\Delta \Gamma_s}{\Gamma_s^L} \cos^2 \left[ \frac{1}{2} \phi_{NP} + \arg C_S \right] \quad \text{SM CP} \quad \frac{\Gamma^H_s}{\Gamma_s^L}$$
Evaluation of the LO Wilson coefficients in the SM:

\[ C_A^{(0)} = \frac{1}{2} Y_0 \left( \frac{m_t^2}{M_W^2} \right), \quad Y_0(x) = \frac{3x^2}{8(x-1)^2} \ln x + \frac{x^2 - 4x}{8(x-1)}, \]

\[ C_{S,P} = \mathcal{O} \left( \frac{m_{W_s}}{M_W} \right). \]

Effects of \( C_{S,P} \) are on the branching ratio are suppressed by \( M_{B_s}^2/M_W^2 \Rightarrow \) negligible.

Thus, only \( C_A \) matters in the SM.
Evaluation of the Wilson coefficients beyond the SM.

Example 1: the Two-Higgs-Doublet Model II

\[ \tan \beta = \frac{v_2}{v_1}, \quad z = \frac{M_{H^\pm}^2}{m_t^2}, \]

\[ C_S \simeq C_P \simeq \frac{m_\mu m_b \tan^2 \beta}{4M_W^2} \frac{\ln z}{z-1} > 0, \]

\[ B(B_s \to \mu^+\mu^-) \simeq \text{(const.)} \left[ \frac{2m_\mu}{M_{B_s}} C_A - C_P \right]^2 + |C_S|^2 \]

\[ C_A = C_A^{SM} + \Delta C_A \]

\[
\begin{align*}
\text{positive} & \quad \text{small} \\
\Rightarrow & \quad \{ \text{suppression for moderate } C_{S,P} \text{, enhancement for huge } \tan \beta \text{ only} \}
\end{align*}
\]
For $M_{H^\pm} = 500 \text{ GeV}$ and $\tan \beta = 50$: suppression by a factor of $\sim 2$. Enhancement possible only for $\tan \beta > 65$. 

[B \to X_s \gamma, 

Evaluation of the Wilson coefficients beyond the SM.

Example 2: the MSSM.

For large $\tan \beta$:

$$B(B_S \to \mu^+ \mu^-) \sim \frac{m_b^2 m_{\mu}^2}{M_A^4} \tan^6 \beta$$

Examples of constraints on the MSSM parameter space:


- **green lines** — bounds from $B_s \rightarrow \mu^+\mu^-$ (CMS & LHCb 2013, exclusion to the left)
- **purple lines** — ATLAS 95%CL bounds from $E_T$+ jets
- **green shaded** — excluded by $b \rightarrow s\gamma$
- **brown shaded** — charged LSP
- **pink shaded** — SUSY helps with $g-2$
- **blue strips** — favoured by $\Omega_{DM}$
Evaluation of the NNLO QCD matching corrections in the SM

W-boxes: (1LPI)

Z-penguins: (1LPI)

All the external momenta and light masses have been set to zero $\Rightarrow$ No loop diagrams on the effective theory side.

Subtleties: (i) counterterms with finite parts $\sim \bar{b}_L \mathcal{D}_S L$

(ii) evanescent operators: $E_B = (\bar{b}\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5 s)(\bar{\mu}\gamma^\rho\gamma^\nu\gamma_5 \mu) - 4(\bar{b}\gamma_\alpha\gamma_5 s)(\bar{\mu}\gamma^\alpha\gamma_5 \mu)$

$E_T = \text{Tr} (\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma_5) (\bar{b}\gamma_\nu\gamma_\rho\gamma_\sigma s)(\bar{\mu}\gamma_\alpha\gamma_5 \mu) + 24(\bar{b}\gamma_\alpha\gamma_5 s)(\bar{\mu}\gamma^\alpha\gamma_5 \mu)$

Renormalization of $E_B$

Diagrams generating $E_T$
Perturbative series for the Wilson coefficient at $\mu = \mu_0 \sim m_t, M_W$:

$$C_A(\mu_0) = C_A^{(0)}(\mu_0) + \frac{\alpha_s}{4\pi} C_A^{(1)}(\mu_0) + (\frac{\alpha_s}{4\pi})^2 C_A^{(2)}(\mu_0) + \frac{\alpha_s}{4\pi} \Delta_{EW} C_A(\mu_0) + \ldots$$

The top quark mass is $\overline{\text{MS}}$-renormalized at $\mu_0$ with respect to QCD, and on shell with respect to the EW interactions. Both $\alpha_s$ and $\alpha_{em}$ are $\overline{\text{MS}}$-renormalized at $\mu_0$ in the effective theory.

To deal with single-scale tadpole integrals, we expand around $y = 1$ (solid lines) and around $y = 0$ (dashed lines), where $y = M_W/m_t$.

The expansions reach $(1 - y^2)^{16}$ and $y^{12}$, respectively. The blue band indicates the physical region.

Matching scale dependence of $|C_A|^2$ gets significantly reduced. The plot corresponds to $\Delta_{EW} C_A(\mu_0) = 0$.

However, with our conventions for $m_t$ and the global normalization, $\mu_0$-dependence is due to QCD only.

**NNLO fit (with $\Delta_{EW} C_A(\mu_0) = 0$):**

$$C_A = 0.4802 \left( \frac{M_t}{173.1} \right)^{1.52} \left( \frac{\alpha_s(M_Z)}{0.1184} \right)^{-0.09} + \mathcal{O}(\alpha_{em})$$
Evaluation of the NLO EW matching corrections in the SM


Method: similar to the NNLO QCD case. Two-loop integrals with three mass scales are present.

Dependence of the final result on $\mu_0$ in various renormalization schemes (dotted – LO, solid – NLO):

In all the four plots: no QCD corrections to $C_A$ included, $m_t(m_t)$ w.r.t. QCD used.

**OS-2 scheme:** Global normalization factor in $\mathcal{L}_{\text{eff}}$ set to $N = V_{tb}^* V_{ts} G_F^2 M_W^2 / \pi^2$

Masses at the LO renormalized on-shell w.r.t. EW interactions (including $M_W$ in $N$)

Plotted quantity: $-2 C_A G_F^2 M_W^2 / \pi^2$ in GeV$^{-2}$

NLO EW matching correction to the BR: $-3.7\%$

**other schemes:** Global normalization factor in $\mathcal{L}_{\text{eff}}$ set to $4 V_{tb}^* V_{ts} G_F / \sqrt{2}$

At the LO, $\alpha_{em}(\mu_0)$ used

$\overline{\text{MS}}$: Masses and $\sin^2 \theta_W$ renormalized at $\mu_0$

OS-1: Masses as in OS-2, $\sin^2 \theta_W$ on-shell

HY (hybrid): Masses as in OS-2, $\sin^2 \theta_W$ as in $\overline{\text{MS}}$. 
TH uncertainties after including the new QCD and EW corrections


\[ \overline{\mathcal{B}}(B_s \to \mu^+\mu^-)_{\text{SM}} = \frac{G_F^4 M_W^4 m_B^2 M_{B_s}}{8\pi^5} \times \]

\[ \times \left| V_{tb}^* V_{ts} \right|^2 \tau_H^s \left\{ f_{B_s}^2 \left[ Y_0 \left( \frac{m_t^2}{M_W^2} \right) + \mathcal{O}(\alpha_s) + \mathcal{O}(\alpha_{\text{em}}) + \mathcal{O}(\alpha_s^2) \right]^2 + \mathcal{O}(\alpha_{\text{em}}) \right\} \]

\[ = (3.65 \pm 0.23) \times 10^{-9} \quad \text{for} \quad f_{B_s} = 227.7(4.5) \text{ MeV} \quad [\text{FLAG, arXiv:1310.8555}] \]

The \( \mathcal{O}(\alpha_s) \) corrections enhance \( \overline{\mathcal{B}} \) by around +2.2\% when \( m_t(m_t) \) is used at the leading order.


Logarithmically \( \ln \left( \frac{m_t^2}{m_b^2} \right) \) enhanced electromagnetic corrections suppress \( \overline{\mathcal{B}} \) by around −1.5\%.


\[ \overline{\mathcal{B}}(B_s \to \mu^+\mu^-)_{\text{exp}} = (2.9 \pm 0.7) \times 10^{-9} \]

Prospects: ±10\% in 2018, ±7\% in 2021, ...

SM predictions for all the branching ratios \( \mathcal{B}_{q\ell} \equiv \mathcal{B}(B_q \to \ell^+\ell^-) \)


\[
\begin{align*}
\mathcal{B}_{se} \times 10^{14} &= (8.54 \pm 0.13) R_{t\alpha} R_s = 8.54 \pm 0.55, \\
\mathcal{B}_{sm} \times 10^{9} &= (3.65 \pm 0.06) R_{t\alpha} R_s = 3.65 \pm 0.23, \\
\mathcal{B}_{s\tau} \times 10^{7} &= (7.73 \pm 0.12) R_{t\alpha} R_s = 7.73 \pm 0.49, \\
\mathcal{B}_{de} \times 10^{15} &= (2.48 \pm 0.04) R_{t\alpha} R_d = 2.48 \pm 0.21, \\
\mathcal{B}_{d\mu} \times 10^{10} &= (1.06 \pm 0.02) R_{t\alpha} R_d = 1.06 \pm 0.09, \\
\mathcal{B}_{d\tau} \times 10^{8} &= (2.22 \pm 0.04) R_{t\alpha} R_d = 2.22 \pm 0.19,
\end{align*}
\]

(LHCb & CMS : 2.9 \pm 0.7)

(LHCb & CMS : 3.6^{+1.6}_{-1.4})

where

\[
\begin{align*}
R_{t\alpha} &= \left( \frac{M_t}{173.1 \text{ GeV}} \right)^{3.06} \left( \frac{\alpha_s(M_Z)}{0.1184} \right)^{-0.18}, \\
R_s &= \left( \frac{f_{Bs}[\text{MeV}]}{227.7} \right)^2 \left( \frac{|V_{cb}|}{0.0424} \right)^2 \left( \frac{|V_{ts}^*V_{td}/V_{cb}|}{0.980} \right)^2 \frac{\tau_s[H \text{ ps}]}{1.615}, \\
R_d &= \left( \frac{f_{Bd}[\text{MeV}]}{190.5} \right)^2 \left( \frac{|V_{tb}^*V_{td}|}{0.0088} \right)^2 \frac{\tau_d^{av}[\text{ps}]}{1.519}.
\end{align*}
\]

Sources of uncertainties:

<table>
<thead>
<tr>
<th>( f_{Bq} )</th>
<th>CKM</th>
<th>( \tau_q^H )</th>
<th>( M_t )</th>
<th>( \alpha_s )</th>
<th>other</th>
<th>non-param.</th>
<th>( \sum )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{B}_{s\ell} )</td>
<td>4.0%</td>
<td>4.3%</td>
<td>1.3%</td>
<td>1.6%</td>
<td>0.1%</td>
<td>(&lt; 0.1% )</td>
<td>1.5%</td>
</tr>
<tr>
<td>( \mathcal{B}_{d\ell} )</td>
<td>4.5%</td>
<td>6.9%</td>
<td>0.5%</td>
<td>1.6%</td>
<td>0.1%</td>
<td>(&lt; 0.1% )</td>
<td>1.5%</td>
</tr>
</tbody>
</table>

In the case of \( \mathcal{B}_{s\ell} \), the main uncertainty (4.2%) originates from \( |V_{cb}| = 0.0424(9) \) that comes from a recent fit to the inclusive semileptonic data [P. Gambino and C. Schwanda, Phys. Rev. D 89 (2014) 014022].
BACKUP SLIDES
Perturbative expansion of the Wilson coefficients:

\[ C_i(\mu) = C_i^{(0)}(\mu) + \frac{\alpha_s(\mu)}{4\pi} C_i^{(1)}(\mu) + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 C_i^{(2)}(\mu) + \ldots \]

**Branching ratio:**

\[ \mathcal{B}(\bar{B} \rightarrow X_{s\gamma})_{E_{\gamma}>E_0} = \mathcal{B}(\bar{B} \rightarrow X_{c\ell\bar{\nu}})_{\exp} \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{em}}{\pi} \left[ P(E_0) + N(E_0) \right] \]

\[ \frac{\Gamma[b \rightarrow X_{s\gamma}]_{E_{\gamma}>E_0}}{|V_{cb}/V_{ub}|^2} \frac{\Gamma[b \rightarrow X_{u\ell\bar{\nu}}]}{\Gamma[b \rightarrow X_{u\ell\bar{\nu}}]} = \left| \frac{V_{ts}^* V_{tb}}{V_{cb}} \right|^2 \frac{6\alpha_{em}}{\pi} P(E_0), \quad C = \left| \frac{V_{ub}}{V_{cb}} \right|^2 \frac{\Gamma[\bar{B} \rightarrow X_{c\ell\bar{\nu}}]}{\Gamma[\bar{B} \rightarrow X_{u\ell\bar{\nu}}]} \]

\[ P(E_0) = \sum_{i,j} C_i C_j K_{ij} \]

Perturbative expansion of \( K_{ij} \):

\[ K_{ij} = K_{ij}^{(0)} + \frac{\alpha_s(\mu_b)}{4\pi} K_{ij}^{(1)} + \left( \frac{\alpha_s(\mu_b)}{4\pi} \right)^2 K_{ij}^{(2)} + \ldots \]

\( \mu_b \sim \frac{m_b}{2} \)

Perturbative expansion of \( P(E_0) \):

\[ P = P^{(0)} + \frac{\alpha_s}{4\pi} \left( P_1^{(1)} + P_2^{(1)}(r) \right) + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \left( P_1^{(2)} + P_2^{(2)}(r) + P_3^{(2)}(r) \right) \]

\[ P_1^{(1)}, P_3^{(2)} \sim C_i^{(0)} C_j^{(1)}, \quad P_2^{(1)}, P_2^{(2)} \sim C_i^{(0)} C_j^{(0)}, \quad P_1^{(2)} \sim (C_i^{(0)} C_j^{(2)}, C_i^{(1)} C_j^{(1)}) \]

\[ r = \frac{m_c}{m_b} \]

Most important at the NNLO: \( K_{77}^{(2)}, K_{27}^{(2)} \) and \( K_{17}^{(2)} \).
Perturbative evaluation of $\Gamma(b \to X_s^p \gamma)$ at $\mu_b \sim \frac{m_b}{2}$.

$$\Gamma(b \to X_s^p \gamma)_{E_\gamma > E_0} = \frac{G_F^2 m_b^5 \alpha_{em}}{32\pi^4} |V_{ts}^* V_{tb}|^2 \sum_{i,j=1}^{8} C_i(\mu_b) C_j(\mu_b) G_{ij}(E_0, \mu_b)$$

**LO:** $G_{77} = 1$ $\iff$ $\gamma$ 

**NLO:** 1996: Quasi-complete $G_{ij}$ $\{\begin{array}{l} \text{[Greub, Hurth, Wyler, 1996]} \\ \text{[Ali, Greub, 1991-1995]} \end{array}$

2002: Complete(*) $G_{ij}$ $\{\begin{array}{l} \text{[Buras, Czarnecki, Urban, MM, 2002]} \\ \text{[Pott, 1995]} \end{array}$

(*) Up to $b \to sq\bar{q}\gamma$ channel contributions involving diagrams similar to the above LO one.

They get suppressed by $\alpha_s C_{3,4,5,6}$ and phase-space for $E_0 \sim m_b/3$.

**NNLO:** We are still on the way to the quasi-complete case:

$G_{77}$ is fully known: $\{\begin{array}{l} \text{[Blokland et al., 2005]} \\ \text{[Melnikov, Mitov, 2005]} \\ \text{[Asatrian et al., 2006-2007]} \end{array}$

$G_{78}$ is fully known: $\{\begin{array}{l} \text{[Asatrian et al., arXiv:1005.5587]} \end{array}$
The most troublesome NNLO contribution to $G_{ij}$:

$G_{27}$: (and analogous $G_{17}$)

$m_c = 0$: Czakon, Fiedler, Huber, MM, Schutzmeier, Steinhauser, to be published

163 massive 4-loop on-shell master integrals (with cuts).

The $m_c \gg m_b/2$ limit is known [Steinhauser, MM, 2006].

The BLM approximation is known for arbitrary $m_c$: \{[Bieri, Greub, Steinhauser, 2003], [Ligeti, Luke, Manohar, Wise, 1999]\}.

Towards $G_{27}$ at the NNLO for arbitrary $m_c$.


1. Generation of diagrams and performing the Dirac algebra to express everything in terms of four-loop two-scale scalar integrals with unitarity cuts.

2. Reduction to master integrals with the help of Integration By Parts (IBP).


The IBP for 2-particle cuts has just been completed with the help of FIRE: $\sim 0.5$ TB RAM has been used $\sim 1$ month at CERN and KIT.

Number of master integrals: around 500.
3. Extending the set of master integrals $I_n$ so that it closes under differentiation with respect to $z = m_c^2/m_b^2$. This way one obtains a system of differential equations

$$\frac{d}{dz} I_n = \sum_k w_{nk}(z, \epsilon) I_k,$$

where $w_{nk}$ are rational functions of their arguments.

4. Calculating boundary conditions for (*) using automatized asymptotic expansions at $m_c \gg m_b$.


This algorithm has already been successfully applied for diagrams with (massless and massive) quark loops on the gluon lines where $18 + 47 + 38 = 103$ master integrals were present.

Non-perturbative contributions from the photonic dipole operator alone ("77" term) are well controlled for $E_0 = 1.6$ GeV:

$$O\left(\frac{\alpha_s \Lambda}{m_b}\right)_{n=0,1,2,...} \text{ vanish, } \quad O\left(\frac{\Lambda^2}{m_b^2}\right)_{\text{Bigi, Blok, Shifman, Uraltsev, Vainshtein, 1992], \quad O\left(\frac{\Lambda^3}{m_b^3}\right)_{\text{Bauer, 1997], \quad O\left(\frac{\alpha_s \Lambda^2}{m_b^2}\right)_{\text{Ewerth, Gambino, Nandi, 2009].}}\right.$$

The dominant non-perturbative uncertainty originates from the "27" interference term:

$$\frac{\Delta B}{B} = -\frac{6C_2 - C_1}{54C_7} \left[ \frac{\lambda_2}{m_c^2} + \sum_n b_n O\left(\frac{\Lambda^2}{m_c^2} \left(\frac{m_b \Lambda}{m_c^2}\right)^n\right) \right]$$

$$\lambda_2 \simeq 0.12 \text{ GeV}^2$$

from $B-B^*$ mass splitting

The coefficients $b_n$ decrease fast with $n$.

[Voloshin, 1996], [Khodjamirian, Rückl, Stoll, Wyler, 1997]
[Grant, Morgan, Nussinov, Peccei, 1997]

New claims by Benzke, Lee, Neubert and Paz in arXiv:1003.5012:

One cannot really expand in $m_b \Lambda / m_c^2$. All such corrections should be treated as $\Lambda / m_b$ ones and estimated using models of subleading shape functions. Dominant contributions to the estimated $\pm 5\%$ non-perturbative uncertainty in $B$ are found this way, with the help of alternating-sign shape functions that undergo weaker suppression at large gluon momenta.

New claims by Benzke, Lee, Neubert and Paz in arXiv:1003.5012:

One cannot really expand in $m_b \Lambda / m_c^2$. All such corrections should be treated as $\Lambda / m_b$ ones and estimated using models of subleading shape functions. Dominant contributions to the estimated $\pm 5\%$ non-perturbative uncertainty in $B$ are found this way, with the help of alternating-sign shape functions that undergo weaker suppression at large gluon momenta.

Correction to the above

Phase-space suppressed

$O\left(\frac{\alpha_s \Lambda}{m_b}\right)$ Main worry in hep-ph/0609232, and reason for the $\pm 5\%$ non-perturbative uncertainty.
Energetic photon production in charmless decays of the $\bar{B}$-meson
\((E_\gamma \gtrsim \frac{m_b}{3} \simeq 1.6 \text{ GeV})\)

[see MM, arXiv:0911.1651]

**A. Without long-distance charm loops:**

1. Hard

```
| s |
```

Dominant, well-controlled.

\(\mathcal{O}(\alpha_s \Lambda/m_b)\), \((-1.6 \pm 1.2)\%\).

[Voloshin, 1996], [...],


[Benzke, Lee, Neubert, Paz, 2010]

2. Conversion

```
| s |
```

\(\sim -0.2\%\) or \((+0.8 \pm 1.1)\%\).

[Capustin, Ligeti, Politzer, 1995]

[Benzke, Lee, Neubert, Paz, 2010]

3. Collinear

```
| s |
```

Exp. \(\pi^0, \eta, \eta', \omega\) subtracted.

Perturbatively \(\sim 0.1\%\).

4. Annihilation \((q\bar{q} \neq c\bar{c})\)

```
| q \rightarrow q | s |
```

\(\mathcal{O}(\Lambda^2/m_c^2), \sim +3.1\%\).

[Voloshin, 1996], [...],


[Benzke, Lee, Neubert, Paz, 2010]


**B. With long-distance charm loops:**

5. Soft gluons only

```
| s |
```

\(\mathcal{O}(\Lambda^2/m_c^2)\), \sim +3.1\%.

[e.g. \(\eta_c, J/\psi, \psi'\)]

6. Boosted light \(c\bar{c}\) state annihilation

```
| s |
```

Exp. \(J/\psi\) subtracted (< 1\%).

Perturbatively (including hard): \sim +3.6\%.

7. Annihilation of \(c\bar{c}\) in a heavy \((\bar{c}s)(\bar{q}c)\) state

```
| s |
```

\(\mathcal{O}(\alpha_s \Lambda^2/M^2)\)

\(M \sim 2m_c, 2E_{\gamma}, m_b\).

\[\text{e.g. } \mathcal{B}[B^- \rightarrow D_sJ(2457)^- D^*(2007)^0 ] \simeq 1.2\%\],

\[\mathcal{B}[B^0 \rightarrow D^*(2010)+ \bar{D}^*(2007)^0 K^-] \simeq 1.2\%\].
The direct CP asymmetry

\[ A_{Xs\gamma} = \frac{\Gamma(\bar{B} \to X_s\gamma) - \Gamma(B \to X_s\bar{\gamma})}{\Gamma(B \to X_s\gamma) + \Gamma(B \to X_s\bar{\gamma})} \]

Semi inclusive measurements \(\Rightarrow A_{Xs\gamma}^{\text{exp}} = -(1.2 \pm 2.8)\% \) (HFAG average)


\[ A_{Xs\gamma}^{\text{SM}} \simeq \text{Im} \left( \frac{V_{us}^* V_{ub}}{V_{ts}^* V_{tb}} \right) \pi \left| \frac{C_1^{\text{their}}}{C_7} \right| \left[ \frac{\Lambda_{17}^{u} - \Lambda_{17}^{c}}{m_b} + \frac{40\alpha_s m_c^2}{9\pi m_b^2} \left( 1 - \frac{2}{5} \ln \frac{m_b}{m_c} + \frac{4}{5} \ln^2 \frac{m_b}{m_c} - \frac{\pi^2}{15} \right) \right] \]

\[ \simeq \left( 1.15 \frac{\Lambda_{17}^{u} - \Lambda_{17}^{c}}{300 \text{ MeV}} + 0.71 \right) \% \in [-0.6\%, +2.8\%] \text{ using } \left\{ \begin{array}{l} -330 \text{ MeV} < \Lambda_{17}^{u} < +525 \text{ MeV} \\ -9 \text{ MeV} < \Lambda_{17}^{u} < +11 \text{ MeV} \end{array} \right\} \]

Despite the uncertainties, \( A_{Xs\gamma} \) provides constraints on models with non-minimal flavour violation. Such models are also constrained by:

\[ A_{X(s+d)\gamma} = \frac{\Gamma(\bar{B} \to X_{(s+d)\gamma}) - \Gamma(B \to X_{(s+d)\bar{\gamma}})}{\Gamma(B \to X_{(s+d)\gamma}) + \Gamma(B \to X_{(s+d)\bar{\gamma}})} \]

\[ (A_{X(s+d)\gamma}^{\text{SM}} \simeq 0) \]