κ-deformed Bohlin-Sundman transformation

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Plan Of The Talk

- Motivation
- Introduction
- Dynamics in κ-space-time
- Conclusion

Why non-commutativity?

- Attempts to localize with extreme precision cause gravitational collapse so that space-time below the Planck scale has no operational meaning.
- Physics at very high energy scale $\Rightarrow$ Minimal length.
- Concept of minimum length leads to introduction of non-commutativity. (S. Doplicher, K. Fredenhagen, J. Roberts, Commun. Math. Phys. 172, 187-220 (1995))
- Some instances of non-commutativity:
  - Landau problem.

$$[x, y] = -i\frac{\hbar c}{eB}$$

    (G. Magro, arXiv:quant-ph/0302001)
  - String theory. (N. Seiberg, E. Witten; JHEP, 9909 **032** 1999)

What is non-commutativity?

- $[x_\mu, x_\nu] = C_{\mu\nu}(x)$.
- Two special scenarios
  1. $[x_i, x_j] \neq 0 \rightarrow$ destroys isotropy of space.
  2. $[x_i, x_0] \neq 0 \rightarrow$ effect boost transformations.
- Consider a space-time: $[x_i, x_j] = 0$ & $[x_0, x_i] = \frac{1}{\kappa} x_i = a x_i$.

This is the definition of a κ-space-time.

Kappa space-time

- Approach: non-commutative $\rightarrow$ commutative.
- $\hat{x}^\mu = x^\mu + \alpha x^\mu (a.p) + \beta (a.x) p^\mu + \gamma a^\mu (x.p)$.
- $\hat{p}^\mu = p^\mu + (\alpha + \beta)(a.p) p^\mu + \gamma a^\mu (p.p)$.
- $\hat{x}^\mu$ and $\hat{p}^\mu$ satisfy the following relations:

$$\begin{aligned}
\{\hat{x}^\mu, \hat{x}^\nu\} &= a^\mu \hat{x}^\nu - a^\nu \hat{x}^\mu, \\
\{\hat{p}^\mu, \hat{p}^\nu\} &= 0, \\
\{\hat{p}^\mu, \hat{x}^\nu\} &= \eta^{\mu\nu}(1 + s(a.p) + (s+2) a^\mu p^\nu + (s+1) a^\nu p^\mu.
\end{aligned}$$

where $s = 2\alpha + \beta$. a^μ is the deformation parameter, α, β and γ are constants with $\gamma - \alpha = 1$, $\beta \in \mathbb{R}$.
(E. Harikumar, T. Jurić, and S. Meljanac; Phys. Rev. **D 84**, 085020 (2011))

Central potentials in commutative space-time

Two familiar examples

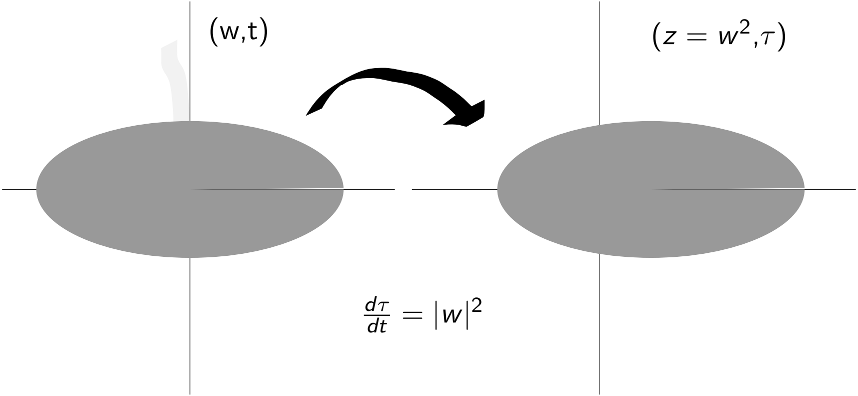
- The Kepler(Coulomb) Potential: $V = \frac{C}{r}$.
- Harmonic Oscillator Potential: $V = -kr$.

Some Nice Properties

- They are only central forces with stable closed circular orbits (Bertrand's Theorem).
- Elliptical path is a solution for both systems.
- Both possess a conserved quantity which is solely property and not corresponding to space-time symmetry.

Question: Does there exist a relation between these problems?
Answer: YES! The answer lies in Bohlin-Sundman transformation.

Bohlin-Sundman transformation



$$\frac{d\tau}{dt} = |w|^2$$

Kinetic term in κ space-time

- $\hat{p}_0 = p_0 + a\alpha {p_0}^2 + a\gamma m^2$
- $\hat{p}_0 \equiv \hat{E} = E_0 + aE_1 + a^2 E_2 + \ldots$

Equating the expressions term by term, one obtains

- $p_0 = E_0$
- $E_1 = (\alpha + \beta) p_0^2 + \gamma m^2$.
- all higher order terms are identically zero.
- $\hat{E} = \frac{p^2}{2\tilde{m}} + m$
- $\tilde{m} = \frac{m}{1+2a\alpha m}$
- We make the choice : $a^\mu = (a, \vec{0})$.
- set $\beta = 0$ to avoid explicit time dependence.

Central potential in κ space-time

- $\hat{x}^{\mu} = x^i + \alpha x^i (aE_0)$
- $V(\hat{x}^i) = U(x) + a\alpha E_0 \frac{dU}{dx} + \frac{(a\alpha E_0)^2}{2!} \frac{d^2 U}{dx^2} + O(a^3)$.
- We make the choice : $a^{\mu} = (a, \vec{0})$.
- set $\beta = 0$ to avoid explicit time dependence.
- In case of power potentials, we have

$$
\begin{aligned}
V(\hat{x}) &= C(x(1 + a\alpha E_0))^n \\
&= \tilde{C} U(r)
\end{aligned}
$$

where $\tilde{C} = C(1 + a\alpha E_0)^n$ and $U(r) = x^n$.

Kepler Potential and Isotropic Oscillator in Two Dimensional κ-Space-Time

Isotropic Oscillator

- $\tilde{V}(\hat{x}^\mu) = \frac{k}{2}\hat{r}^2$
- $\tilde{V}(r) = \frac{k}{2}(1 + 2a\alpha E_0)r^2$
- $H = \frac{p_i^2}{2\tilde{m}} + \frac{\tilde{k}}{2}r^2$
- $\tilde{k} = k(1 + 2a\alpha E_0)^2$
- $\tilde{m}\ddot{\vec{r}} = -\tilde{k}\vec{r}$
- $\tilde{m} = \frac{m}{1+2a\alpha m}$

Kepler Potential

- $\tilde{V}(\hat{x}^\mu) = -\frac{C}{\hat{r}}.$
- $\tilde{V}(r) = -\frac{C}{r}(1 + a\alpha E_0)^{-1}$
- $H = \frac{p_i^2}{2\tilde{m}} - \frac{\tilde{C}}{r}$
- $\tilde{C} = C\frac{1}{(1+a\alpha E_0)}$
- $\ddot{\vec{r}} + \frac{\tilde{C}}{\tilde{m}}\frac{\vec{r}}{r^3} = 0$
- $\tilde{m} = \frac{m}{1+2a\alpha m}$

Potentials in complex coordinates

Notation :$z = z_1 + iz_2$ for Kepler potential and $w = w_1 + iw_2$ for Oscillator potential.

Isotropic Oscillator

- $E_{H.O.} = \frac{\tilde{m}}{2}\dot{w}^2 + \frac{\tilde{k}}{2}|2|^2$
- $\mathcal{F} = \frac{\tilde{m}}{2}\dot{w}^2 + \frac{\tilde{k}}{2}w^2$
- $\tilde{m}\ddot{w} = -\tilde{k}w$

Kepler Potential

- $E_K = \frac{\tilde{m}}{2}|z'|^2 - \frac{\tilde{C}}{|z|}$
- $i\tilde{L} = \tilde{m}(\bar{z}z' - z\bar{z}')$
- $\tilde{A}/\tilde{m} = i\tilde{L}z' + \tilde{C}\frac{z'}{|z|}$
- $\ddot{z} + \frac{\tilde{C}}{\tilde{m}}\frac{z}{r^3} = 0$

The Fradkin-Hill tensor,F_{ij} in commutative space, has the form

$$F = \begin{pmatrix} F_{11} & F_{12} \\ F_{12} & F_{22} \end{pmatrix} = \begin{pmatrix} \frac{m}{2}\left(\dot{x}\right)^2 + \frac{k}{2}x^2 & m\frac{\dot{x}\dot{y}}{2} + k\frac{xy}{2} \\ m\frac{\dot{x}\dot{y}}{2} + k\frac{xy}{2} & \frac{m}{2}\left(\dot{y}\right)^2 + \frac{k}{2}y^2 \end{pmatrix}$$

Duality of Force law

Under the change of variables, $z = w^2$ and $\frac{d\tau}{dt} = |w|^2$.

$$\begin{aligned} \mathcal{F} &= \frac{\tilde{m}}{8} z'(\bar{z}z' - z\bar{z}') + \tilde{C}\frac{z}{4|z|} \\ &= \frac{1}{4}(i\tilde{L}z' + \tilde{C}\frac{z'}{|z|}) \\ &= -\tilde{A}/4\tilde{m}. \end{aligned}$$

where we have used the relation, $E_K = -2\tilde{k}$ which is valid for this dual motion.

We thus have shown that the **conservation of Laplace-Runge-Lenz vector implies the conservation of Fradkin-Hill-Jauch tensor under the Bohlin-Sundman map**.

Conclusion

- We have extended the Bohlin-Sundman transformation in κ-deformed transformation.
- The known duality in commutative space-time between Kepler potential and two- dimensional isotropic potential is extended to the case of κ-deformed space-time.
- Kepler potential and two-dimensional isotropic oscillator potential are super integrable in κ space-time.

(Partha Guha, E. Harikumar and N. S. Z., arXiv:1404.6321).

Thank You !